ELEMENTS

OF

INFINITESIMAL CALCULUS.

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PREFACE.

THE present treatise, as its title points out, has been based on the principles of the infinitesimal method. I think that this method, besides being the simplest of all for teaching Calculus, as all admit, is also the most consistent and philosophical. Some modern writers, indeed, have tried to discredit it; but, when we examine their reasonings, we soon discover that they have done so because they have failed to grasp the true nature of infinitesimal quantities. This I have endeavored to show in my introduction to this work, where the reader will find not only what I believe to be the exact notion of the infinitesimal, but also a hint at the grounds (elsewhere developed) upon which the infinitesimal method rests its claim to be preferred to its fashionable rival, the method of limits.

This work being intended for young men who are supposed to devote a considerable part of their time to the study of mental and of natural philosophy, it has been necessary to limit its developments by giving less prominence to the analytical than to the practical portion of it. But, while anxious not to overtax our young people, I have nevertheless collected and condensed all that seemed to be of practical use in this branch of study; and, though I have made it a point to be concise, I have constantly endeavored to make all things as plain as the subject matter permitted. I hope that the average student will need only a moderate effort to understand the object and the processes of differentiation and of integration as laid down in this treatise; though, as to integration, his success will often depend also on his endurance of analytical work. To assist him in the performance of this task, I have developed a sufficient number of geometrical and of mechanical problems, together with some fundamental notions of rational mechanics, which were indispensable, and which will serve as an introduction to the regular study of this latter science.

J. B.

CONTENTS.

INFINITESIMAL CALCULUS.

INTRODUCTION.

1. The Infinitesimal Calculus is exclusively concerned with continuous quantities; for these alone admit of *infinitesimal* variations. A variable quantity is said to be *continuous*, when it is of such a nature that it cannot pass from one value to another without passing through all the intermediate values. All the parts of a continuous quantity are continuous: and, as all continuum is divisible, every part of a continuous quantity, how small soever it be, is still further divisible. In other terms, the division of continuum can have no end.

2. Infinitesimal quantities are sometimes conceived as resulting from an endless division of the finite. But this is not the real genesis of infinitesimals; for, in the order of nature, it is the infinitesimal itself that gives origin to the finite. Thus, an infinitesimal instant of duration does not arise from any division of time; for it is the instant itself that by its flowing generates time. In like manner, the infinitesimal length described by a moving point in one instant of time does not originate in any division of length; for it is the actual infinitesimal motion of the point itself that by its continuation generates a finite length in space.

Hence infinitesimals of time and of length are not mathematical fictions. They are true objective realities. Had they not a real existence in nature, neither the origin nor the variations of continuous movement would be conceivable.

For the same reason we must admit that continuous action cannot produce acceleration except by communicating at every instant of time an infinitesimal degree of velocity: and speaking generally, *all continuous quantities develop by infinitesimal moments.* Hence the branch of Mathematics which investigates the relations between the continuous developments of variable quantities, has received the name of *Infinitesimal Calculus*, and its method of investigation has been called the *infinitesimal method.*

This method has been used by the best mathematicians up to recent times. Poisson, in the introduction to his classical *Traité de Mécanique* (n. 12), says: "In this work I shall exclusively use the infinitesimal method. . . . We are necessarily led to the conception of infinitesimals when we consider the successive variations of a magnitude subject to the law of continuity. Thus time increases by degrees less than any interval that can be assigned, however small it may be. The spaces measured by the various points of a moving body increase also by infinitesimals; for no point can pass from one position to another without traversing all the intermediate positions, and no distance, how small soever, can be assigned between two consecutive positions. Infinitesimals have, then, a real existence: they are not a mere conception of Mathematicians."

3. Modern authors often define the infinitesimal as *the limit of a decreasing quantity.* This definition we cannot approve. For the divisibility of continuum has no limit, and therefore cannot lead to a limit. This is so true, that even those authors confess that the limit—the absolute zero—can never be reached. On the other hand, infinitesimals, in the order of nature, do not arise from finite quantities: it is, on the contrary, these quantities that arise from them. The origin of infinitesimals is dynamical; for they essentially either consist in, or depend on, motion: and as motion has no other being than its actual becoming or developing, so also infinitesimals have but the fleeting existence of the instant in which they become actual. It is for this reason that Sir Isaac Newton conceived them as *fluxions* and *nascent quantities;* that is, quantities not yet developed, but in the very act of developing. This is, we believe, the true notion of the infinitesimal, the only one calculated to satisfy a philosophical mind.* So long as it remains true that a line cannot be drawn except by the motion of a point, so long will it remain true that an infinitesimal line is the *fluxion* of a point through two consecutive positions.

An infinitesimal change may be defined, a change which is brought about in an instant of time. Now, the true instant is *the link of two consecutive terms of duration:* and it is obvious that between two *consecutive* terms of duration there is no room for any *assignable* length. Hence the fleeting instant has a duration *less than any assignable*

* On the modern doctrine and method of limits see the note appended to No. 23.

length of duration, that is, it has a duration strictly infinitesimal. And in the same manner, every other infinitesimal change is *a link of two consecutive terms*, or *of two consecutive states;* for it takes place in an infinitesimal instant.

4. But here the question arises: How can an infinitesimal quantity be intercepted between two *consecutive* points? Consecutive points touch one another and leave no room between them. It would seem, then, that what we call "an infinitesimal" is not a quantity, but a mere nothing. We answer that a point in motion has always two consecutive modes of being in space; for it is always leaving its last position, and always reaching a following position which cannot but be consecutive to the last abandoned.. Now, it is plain, that if the actual passage from the one to the other were not a real change, the whole movement would be without change; for the whole movement is but a continuous passage through consecutive points. But movement without change is a contradiction. It is therefore necessary to concede that between two consecutive points there is room enough for an infinitesimal change, and accordingly for an infinitesimal quantity.

As a further explanation of this truth, let us conceive two material points moving uniformly, the one with a velocity 1, the other with a velocity 2. Their movement being essentially continuous, there is no single instant in the whole of its duration, in which they do not pass from one point to a consecutive point, the one with its velocity 1, the other with its velocity 2. But the velocity 2 causes a change twice as great as that due to the velocity 1.

Therefore the ratio of the two movements is, *at every instant*, as 2:1. But two absolute nothings cannot be in the ratio 2:1. Therefore the movements comprised between two consecutive points are not mere nothings, but are real quantities, though infinitely small. They are, in fact, *fluxions*, or *nascent quantities*, or, as the Schoolmen would say, quantities *in fieri*.

Nor does it matter that these infinitesimals are sometimes represented by the symbol 0. For this symbol has two meanings in mathematics. When it expresses the result of subtraction, as when we have $a-a=0$, it certainly means an *absolute nothing:* but when it expresses the result of division, it is a *real quotient*, and it always means a quantity less than any assignable quantity: but because it has no value in comparison with finite quantities, it is treated as a *relative nothing*, and is represented by 0. Thus, in the equation

$$\frac{a}{\infty}=0$$

the zero represents an infinitesimal quantity. This can be easily proved. For it is only continuous quantities that admit of being divided in infinitum: and, when so divided, they give rise to none but continuous quotients, because every part of continuum is necessarily continuous. Now, the absolute zero cannot be considered continuous. Therefore the absolute zero can never be the quotient of an endless division. And in this sense, it is true, as the theory of limits affirms, that a decreasing quantity may indefinitely tend to the limit zero, but can never reach it. On the other hand, the above equation gives

$$a = 0 \times \infty;$$

and this does certainly not mean that the finite quantity a is equal to an infinity of absolute nothings.

5. We have said that infinitesimals have *no value* as compared with finite quantities. A few years ago, an American writer* was bold enough to maintain that this fundamental principle of infinitesimal analysis is not correct. The principle, however, has been admitted by the greatest mathematicians, and its correctness will not be doubted by any one who understands the real nature of infinitesimals. The principle, says Poisson (loc. cit.), "consists in this, that two finite quantities which do not differ from each other except by an infinitesimal quantity, must be considered as equal; for *between them no inequality, how small soever, can be assigned*"; because the infinitesimal is less than any assignable quantity.

Again, it is plain that the infinitesimal is to the finite as the finite is to the infinite. Now, the infinite is not modified, as to its value, by the addition of a finite quantity. Therefore the finite is not modified by the addition of an infinitesimal. That the infinite is not modified by the addition of a finite quantity, can be assumed as an evident truth: but it can also be demonstrated. Thus, it is shown in Trigonometry that between the angles A, B, C of a

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* Mr. Albert Taylor Bledsoe in his *Philosophy of Mathematics*, where he strives to prove that the infinitesimal method should be abandoned. We are afraid that philosophical readers will not consider his effort a success.

plane triangle there is the relation

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C;$$

and this equation, taking $A = 90°$, $B = 45°$, $C = 45°$, gives

$$\infty + 2 = \infty;$$

which shows that the addition of a finite quantity does not modify the value of the infinite.

We may draw from Arithmetic a still plainer proof of our principle. Dividing 1 by 3 we obtain

$$\frac{1}{3} = 0.333333 \ldots$$

and multiplying this by 3, we obtain

$$1 = 0.999999 \ldots$$

In this last equation, if the second member be understood to continue without end, the difference between the two members will be an infinitesimal fraction—viz., unity divided by a divisor infinitely great. Now, we can prove, that, *notwithstanding this infinitesimal difference*, the equation is *rigorously* true. For, let the second member of the equation be represented by x; then

$$x = 0.999999 \ldots$$

Multiply both members of this by 10; then

$$10x = 9.999999 \ldots = 9 + x;$$

and from this, by reduction, we have

$$9x = 9, \ x = 1.$$

This clearly shows that the equation $1 = 0.999999 \ldots$ is rigorously true. It is plain, therefore, that *an infinitesimal difference has no bearing on the value of a finite quantity*, and that no error is committed by suppressing an infinitesimal by the side of a finite quantity.

6. The notions above developed may suffice as a first introduction to the infinitesimal calculus. We have shown—

1st. That infinitesimals are not nothings, but objective realities:

2d. That infinitesimals are not limits of decreasing quantities, but fluxions—that is, quantities in the act of developing, or more briefly, nascent quantities, whose value is less than any assignable value of the same nature:

3d. That infinitesimals may have different relative values, and form different ratios:

4th. That an infinitesimal, whether added to, or taken from, a finite quantity, cannot modify its value.

As to the different orders of infinitesimals, of which we shall have to speak throughout our treatise, we have here simply to state the fact, that infinitely great, and infinitely small quantities are capable of degrees, so that there may be infinites and infinitesimals of different orders, each infinite of a higher order being infinitely greater than the infinite of a lower order, and each infinitesimal of a higher order being infinitely less than the infinitesimal of a lower order. How this can be, one may not find easy to explain, because both the infinite and the infinitesimal lie beyond the reach of human comprehension: nevertheless we know, not only from Algebra and Geometry, but also from rational philosophy, that such orders of infinites and of infinitesimals cannot be denied. We know that the species ranges infinitely above the individual, and the genus infinitely above the species. Substance extends infinitely less than Being, animal infinitely less than substance, man infinitely less than animal. From this it will be seen that the notion of an infinite infinitely greater than another infinite, is not a dream of our imagination, but a well-founded philosophical conception, familiar to every student of Logic, and admitted, implicitly at least, by every rational being.

Let us, then, write the following series:

$$\ldots\; x^3,\; x^2,\; x,\; 1,\; \frac{1}{x},\; \frac{1}{x^2},\; \frac{1}{x^3}.\; \ldots$$

If we assume $x=\infty$, it is plain that the first term will be infinitely greater than the second, the second infinitely greater than the third, and so on. The middle term 1 being finite, all the following terms are infinitesimal, and each is infinitely less than the one that precedes it. Hence infinites and infinitesimals are distributed into orders. Thus, if x be an infinite of the first order, x^2 will be of the second order, x^3 of the third, etc.; and in like manner $\frac{1}{x}$ will be an infinitesimal of the first order, $\frac{1}{x^2}$ of the second order, $\frac{1}{x^3}$ of the third, etc.

7. The problems whose solution depends on the infinitesimal calculus, are generally such that their conditions cannot be fully expressed in terms of finite quantities. Hence a method had to be found by which to express such conditions in infinitesimal terms. The part of the Calculus which gives rules for properly determining such infinitesimals and their relations, is called the *Differential Calculus*. As, however, none of such infinitesimals must remain in the final solutions, rules were also to be found for passing from the infinitesimal terms to the finite quantities, of which they are the elements; and to effect this, a second part of Calculus was invented under the name of *Integral Calculus.*

Of these two parts of the infinitesimal calculus we propose to give a substantial outline in the present treatise: and we shall add a sufficient number of exercises concerning the application of the Calculus to the solution of geometric and mechanical questions; for it is by working on particular examples that the student will be enabled to appreciate and utilize the manifold resources of this branch of Mathematics.

PART I.

DIFFERENTIAL CALCULUS.

8. Our object in this part of our treatise is to find, and to interpret, the relations which may exist between the infinitesimal changes of correlated quantities varying according to any given law of continuous development. Such a law is mathematically expressed by an equation between the variable quantities; and it is, therefore, from some such equation that the relative values of the infinitesimal changes must be derived.

An infinitesimal change is usually called a *differential*, because it is the difference between two consecutive values, or states, of a variable quantity. The process by which differentials are derived from given equations is called *differentiation*, and the equations themselves, by the same process, are said to be *differentiated*. Hence this part of infinitesimal analysis received the name of *Differential Calculus*.

Differentials are expressed by prefixing the letter d before the quantities to be differentiated. Thus, $dx =$ differential of x, $d(ax^2) =$ differential of ax^2.

9. When an equation contains only two variables, arbitrary values can be assigned to one of them, and the equation will give the corresponding values of the other. The one to which arbitrary values are assigned is called the *independent* variable, and the other, whose value depends on the value assigned to the first, is said to be a *function* of the same. Thus in the equation of the parabola, $y^2 = 2px$, if we take x as independent, y will be a function of x.

When an equation contains more than two variables, then all the variables but one can receive arbitrary values, and are, therefore, *independent*, whilst the remaining one will be a *function* of all the others.

Functions are often designated as follows:

$$y = f(x),\ z = F(x, y),\ \varphi(x, y, z) = 0\,;$$

the first and second being *explicit* functions, the third *implicit*. Different characteristics, as f, F, φ, denote equations having different forms. In these equations, besides the variables, there are usually some constant quantities, though, for the sake of brevity, the variables alone are expressly pointed out.

This part of our treatise will contain three sections. In the first we shall explain the rules of differentiation for all known functions. In the second we shall consider the successive differentials, and their extensive use in mathematical investigations. In the third we shall show the bearing of differential expressions on the solution of problems regarding curves.

SECTION I.

RULES OF DIFFERENTIATION.

10. The function of which we have to find the differential, may be either algebraic or transcendental. It is *algebraic*, if it is formed of expressions obtained by the *ordinary* operations of algebra, as addition, subtraction, multiplication, division, and the formation of powers with constant exponents, entire or fractional. It is *transcendental* when it contains logarithms, circular functions, or exponentials.

Algebraic Functions.

11. POLYNOMIALS. An algebraic sum of functions constitutes a polynomial. Let

$$y = s - at + bu - v + c \qquad (1)$$

be a polynomial, in which s, t, u, v are functions of x, and a, b, c constant quantities. When x acquires an infinitesimal increment dx, the functions s, t, u, v acquire the corresponding infinitesimal increments ds, dt, du, dv, and y acquires its increment dy. Hence the equation (1) becomes

$$y + dy = s + ds - a(t + dt) + b(u + du) - (v + dv) + c. \qquad (2)$$

Subtracting (1) from (2) we shall have

$$dy = ds - adt + bdu - dv, \qquad (3)$$

which is the differential of the given polynomial.

Comparing (3) with (1), we see, first, that the differential of a polynomial is the algebraic sum of the differentials of its terms: second, that *constant factors* (as a and b) remain unchanged: third, that *isolated constant terms*, as c, disappear in the process of differentiation.

12. PRODUCTS. Let us now have

$$y = st, \qquad (1)$$

s and t being functions of x. When x acquires the increment dx, these functions acquire the corresponding increments ds and dt, and y will become $y + dy$. Hence

$$y + dy = (s + ds)(t + dt). \qquad (2)$$

Subtracting (1) from (2) and reducing, we obtain

$$dy = tds + sdt + dtds.$$

But the term $dtds$, as being an infinitesimal of the second order, has no value by the side of the other terms of the equation, which are of the first order. And therefore, suppressing that term, we shall have simply

$$dy, \text{ or } d(st) = tds + sdt. \qquad (3)$$

Hence, *to differentiate the product of two functions of x, we multiply each by the differential of the other, and take the algebraic sum of the results.*

If in (3) we assume $s = uv$, we obtain by this rule

$$ds = udv + vdu;$$

and this substituted in (3) gives

$$d(uvt) = uvdt + utdv + vtdu;$$

which shows that the differential of the product of three functions is obtained *by multiplying the differential of each function by the product of the other functions, and by taking the algebraic sum of the results.* And the same rule holds when the functions are more than three.

13. Quotients. Assume

$$y = \frac{s}{t}, \tag{1}$$

s and t being functions of x. When x becomes $x + dx$, then s, t, and y become $s + ds$, $t + dt$, and $y + dy$. Accordingly

$$y + dy = \frac{s + ds}{t + dt}. \tag{2}$$

Subtracting (1) from (2) we obtain

$$dy = \frac{s + ds}{t + dt} - \frac{s}{t} = \frac{tds - sdt}{t^2 + tdt}.$$

But the term tdt has no value by the side of t^2. Hence we suppress it, and we obtain

$$dy, \text{ or } d\left(\frac{s}{t}\right) = \frac{tds - sdt}{t^2}. \tag{3}$$

Therefore, the differential of a quotient of two functions of the same variable is equal to *the denominator into the differential of the numerator, minus the numerator into the differential of the denominator, divided by the square of the denominator.*

14. Powers and Roots. Let us have

$$y = s^m, \tag{1}$$

s being a function of x, and m a constant. When x becomes $x + dx$, this equation becomes

$$y + dy = (s + ds)^m,$$

or, developing by the binomial formula,

$$y + dy = s^m + ms^{m-1}\,ds + \tfrac{m(m-1)}{2}s^{m-2}\,ds^2 + \ldots \tag{2}$$

Subtracting (1) from (2), we obtain

$$dy = ms^{m-1}\,ds + \tfrac{m(m-1)}{2}s^{m-2}\,ds^2 + \ldots,$$

which, by suppressing all the terms that transcend the first order, reduces to

$$dy, \text{ or } d(s^m) = ms^{m-1}\,ds. \tag{3}$$

Hence, to obtain the differential of any power of a function, *diminish the exponent by* 1, *and then multiply the result by the original exponent and by the differential of the function.*

As the binomial formula is true for all exponents, whether positive or negative, integral or fractional, equation (3) holds good for all possible powers and roots. Thus, making $m = \frac{1}{n}$, and

$$y = \sqrt[n]{s} = s^{\frac{1}{n}},$$

the differential will be

$$dy = \frac{1}{n}s^{\frac{1}{n}-1}\,ds = \frac{1}{n}s^{\frac{1-n}{n}}\,ds = \frac{1}{n}\,\frac{ds}{s^{\frac{n-1}{n}}} = \frac{ds}{n\sqrt[n]{s^{n-1}}}.$$

If $n = 2$, then

$$d\left(s^{\frac{1}{2}}\right) = d\sqrt{s} = \frac{ds}{2\sqrt{s}},$$

that is, *the differential of the square root of a quantity is equal to the differential of the quantity divided by twice the radical.*

The preceding rules are sufficient for the differentiation of any algebraic function of one variable.

EXAMPLES. It is of the utmost importance that the student should at once familiarize himself with the above rules of differentiation, and test, by examples, his practical knowledge of them. Let him work out the following:

1. $y = ax^3 - bx + ac,\quad dy = (3ax^2 - b)\,dx,$

2. $y = (a^2 + x^2)^2 - b,\quad dy = 4(a^2 + x^2)\,xdx,$

3. $y = ax^2 + \dfrac{b}{x},\quad dy = \left(2ax - \dfrac{b}{x^2}\right)dx,$

4. $y = \dfrac{a+x}{a-x},\quad dy = \dfrac{2adx}{(a-x)^2},$

5. $y = 2ax^3 - 3ax^2,\quad dy = 6ax\,(x-1)\,dx,$

6. $y = \dfrac{ax}{(a-x)^2},\quad dy = \dfrac{a\,(a-x) + 4ax^2}{(a-x)^3}dx,$

7. $y = \dfrac{ax^2 - bx}{x-a},\quad dy = \dfrac{a\,(x^2 - 2ax + b)}{(x-a)^2}dx,$

8. $y = \sqrt{a^2 - x^2},\quad dy = -\dfrac{xdx}{\sqrt{a^2 - x^2}},$

9. $y = \sqrt{a^2 - (b-x)^2},\quad dy = \dfrac{(b-x)\,dx}{\sqrt{a^2 - (b-x)^2}},$

10. $y = x\sqrt{a^2+x^2}$, $\qquad dy = \dfrac{(a^2+2x^2)\,dx}{\sqrt{a^2+x^2}}$,

11. $y = \sqrt{\dfrac{x+1}{x-1}}$, $\qquad dy = -\dfrac{dx}{(x-1)\sqrt{x^2-1}}$,

12. $y = \sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$, $\qquad dy = -\dfrac{2a^2x\,dx}{(a^2-x^2)\sqrt{a^4-x^4}}$,

13. $y = (a^2-x^2)\sqrt{a^2+x^2}$, $dy = -\dfrac{(a^2+3x^2)\,x\,dx}{\sqrt{a^2+x^2}}$,

14. $y = (2ax-x^2)^3$, $\qquad dy = 6(a-x)(2ax-x^2)^2\,dx$,

15. $y = \sqrt{x+\sqrt{a^2+x^2}}$; $dy = \dfrac{\sqrt{x+\sqrt{a^2+x^2}}}{2\sqrt{a^2+x^2}}\,dx$,

16. $y = x(1+x^2)\sqrt{1-x^2}$, $dy = \dfrac{1+x^2+4x^4}{\sqrt{1-x^2}}\,dx$,

17. $y = \left(\dfrac{x}{\sqrt{1-x^2}}\right)^3$, $\qquad dy = \dfrac{3x^2dx}{\sqrt{(1-x^2)^5}}$,

18. $y = \dfrac{a+x}{\sqrt{a-x}}$, $\qquad dy = \dfrac{(3a-x)dx}{2\sqrt{(a-x)^3}}$,

19. $y = \dfrac{x}{x+\sqrt{1-x^2}}$, $\qquad dy = \dfrac{dx}{2x(1-x^2)+\sqrt{1-x^2}}$,

20. $y = \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$, $\qquad dy = \dfrac{dx}{2(1+\sqrt{x})\sqrt{x-x^2}}$.

Note. The differentiation of complex functions is often simplified by the use of auxiliary functions. Thus our twelfth example, by assuming

$$\sqrt{a^2-x^2} = s, \text{ and } \sqrt{a^2+x^2} = t,$$

becomes

$$y = \frac{s}{t}, \text{ whence } dy = \frac{tds - sdt}{t^2}.$$

On the other hand, the differentials of the auxiliary functions are

$$ds = -\frac{xdx}{\sqrt{a^2 - x^2}}, \; dt = \frac{xdx}{\sqrt{a^2 + x^2}};$$

and these values, and those of s and t, substituted in the expression for dy, give the differential in terms of x.

Transcendental Functions.

15. Logarithmic Functions. Let s be a function of x, and let

$$y = \log s. \tag{1}$$

When x receives the increment dx, this equation becomes

$$y + dy = \log(s + ds). \tag{2}$$

Subtracting (1) from (2), we have

$$dy = \log(s + ds) - \log s = \log \frac{s + ds}{s} = \log\left(1 + \frac{ds}{s}\right)$$

But we have from Algebra

$$\log\left(1 + \frac{ds}{s}\right) = M\left(\frac{ds}{s} - \frac{1}{2}\left(\frac{ds}{s}\right)^2 + \frac{1}{3}\left(\frac{ds}{s}\right)^3 \ldots\right)$$

Hence, substituting, and suppressing all the infinitesimals of the second and higher orders, we shall have

$$dy, \text{ or } d.(\log s) = M\frac{ds}{s}. \tag{3}$$

The factor M is the modulus of the system of logarithms. In the common system, whose base is 10, we have $M = 0.43429448\ \dots$ In the Napierian system, whose base is

$$e = 2.718281828459\ \dots$$

we have $M = 1$. The logarithms used in the Calculus are always Napierian, or hyperbolic, if no warning be given to the contrary; and the differential of the logarithm becomes simply

$$d(\log s) = \frac{ds}{s}.$$

16. Exponential Functions. Let a be a constant quantity, and s any function of x. The exponential

$$y = a^s \qquad (1)$$

will be readily differentiated by the following process. Take the logarithm of both members of (1), differentiate, and reduce. Thus,

$$\log y = s \log a, \quad \frac{dy}{y} = \log a . ds,$$

whence $dy = y \log a . ds$, or

$$d(a^s) = \log a . a^s ds.$$

If a be changed into e, then, since $\log e = 1$, we have simply

$$d(e^s) = e^s ds.$$

EXAMPLES.

1. $y = \log (a + bx)$, $\qquad dy = \dfrac{bdx}{a + bx}$,

2. $y = \log(a - bx)$, $\qquad dy = -\dfrac{bdx}{a - bx}$,

3. $y = \log\dfrac{a + x}{a - x}$, $\qquad dy = \dfrac{2adx}{a^2 - x^2}$,

4. $y = \log\sqrt{1 + x}$, $\qquad dy = \dfrac{1}{2}\dfrac{dx}{1 + x}$,

5. $y = \log\sqrt{\dfrac{1 + x}{1 - x}}$, $\qquad dy = \dfrac{dx}{1 - x^2}$,

6. $y = \log(x + a + \sqrt{2ax + x^2})$, $\qquad dy = \dfrac{dx}{\sqrt{2ax + x^2}}$,

7. $y = \log(x + \sqrt{x^2 + a^2})$, $\qquad dy = \dfrac{dx}{\sqrt{x^2 + a^2}}$,

8. $y = \log\dfrac{x}{\sqrt{1 - x^2}}$, $\qquad dy = \dfrac{dx}{x(1 - x^2)}$,

9. $y = x\log x$, $\qquad dy = (1 + \log x)\,dx$,

10. $y = \log(\log x)$, $\qquad dy = \dfrac{dx}{x\log x}$,

11. $y = e^x(x - 1)$, $\qquad dy = e^x x dx$,

12. $y = e^x\log x$, $\qquad dy = e^x\left(\log x + \dfrac{1}{x}\right)dx$,

13. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, $\qquad dy = \dfrac{4dx}{(e^x + e^{-x})^2}$,

14. $y = x^x$, $\qquad dy = x^x(1 + \log x)dx$,

15. $y = \log(e^x + e^{-x})$, $\qquad dy = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}dx$.

17. TRIGONOMETRIC FUNCTIONS. *Differential of sine.* Let s be a function of x. The equation

$$y = \sin s, \tag{1}$$

when x receives the increment dx, becomes

$$y + dy = \sin(s + ds), \tag{2}$$

and subtracting (1) from (2),

$$dy = \sin(s + ds) - \sin s$$
$$= \sin s \cos ds + \cos s \sin ds - \sin s.$$

But $\cos ds = 1$, and $\sin ds = ds$. Substituting, and reducing,

$$dy = d(\sin s) = \cos s\, ds.$$

Differential of cosine. When $y = \cos s$, we shall find

$$dy = \cos(s + ds) - \cos s$$
$$= \cos s \cos ds - \sin s \sin ds - \cos s;$$

which, because $\cos ds = 1$, and $\sin ds = ds$, reduces to

$$dy = d(\cos s) = -\sin s\, ds.$$

Differential of tangent. When $y = \tan s$, we shall have

$$dy = d\left(\frac{\sin s}{\cos s}\right) = \frac{\cos s\, d(\sin s) - \sin s\, d(\cos s)}{\cos^2 s},$$

that is,

$$dy = \frac{\cos^2 s + \sin^2 s}{\cos^2 s} ds,$$

or, reducing,

$$dy = d\,(\tan s) = \frac{ds}{\cos^2 s}.$$

Differential of cotangent. When $y = \cot s$, we shall have

$$dy = d\left(\frac{\cos s}{\sin s}\right) = \frac{\sin s\, d\,(\cos s) - \cos s\, d\,(\sin s)}{\sin^2 s},$$

that is,

$$dy = -\frac{\cos^2 s + \sin^2 s}{\sin^2 s}\, ds,$$

or, reducing,

$$dy = d\,(\cot s) = -\frac{ds}{\sin^2 s}.$$

Differential of secant. If $y = \sec s = \frac{1}{\cos s}$, we shall have

$$dy = d\,(\sec s) = \frac{\sin s\, ds}{\cos^2 s}.$$

Differential of cosecant. If $y = \text{cosec}\, s = \frac{1}{\sin s}$, we shall have

$$dy = d\,(\text{cosec}\, s) = -\frac{\cos s\, ds}{\sin^2 s}.$$

Differential of versed-sine. If $y = \text{vers}\ s = 1 - \cos s$, then

$$dy = d\,(\text{vers}\, s) = \sin s\, ds$$

Differential of versed-cosine. If $y = \text{covers } s = 1 - \sin s$, then

$$dy = d\,(\text{covers } s) = -\cos s\, ds.^*$$

EXAMPLES.

1. $y = \sin \frac{m}{n} x, \qquad dy = \frac{m}{n} \cos \frac{m}{n} x dx,$

2. $y = \sin^3 x, \qquad dy = 3 \sin^2 x \cos x dx,$

3. $y = \sin nx \cos nx, \; dy = n\,(\cos^2 nx - \sin^2 nx)\, dx,$

4. $y = \log \sin(a - x), \; dy = -\cot\,(a - x)\, dx,$

5. $y = \sin x - x \cos x, \; dy = x \sin x dx,$

6. $y = \tan \sqrt{1 - x}, \; dy = -\frac{dx \sec^2 \sqrt{1 - x}}{2\sqrt{1 - x}},$

7. $y = \log \tan \left(\frac{\pi}{4} - \frac{x}{2}\right), \; dy = -\frac{dx}{\cos x},$

8. $y = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4}, \; dy = \sin^3 x \cos^3 x dx,$

9. $y = \log \frac{a + b \tan \frac{1}{2} x}{a - b \tan \frac{1}{2} x}, \; dy = \frac{ab.dx}{a^2 \cos^2 \frac{1}{2} x - b^2 \sin^2 \frac{1}{2} x},$

10. $y = \tan x - \cot x, \; dy = \frac{dx}{\sin^2 x \cos^2 x},$

11. $y = e^{\cos x} \sin x, \qquad dy = e^{\cos x} (\cos x - \sin^2 x)\, dx,$

12. $y = \log\,(\cos x - \sqrt{-1} \sin x), \; dy = dx \sqrt{-1}.$

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* In all these equations s represents the length of the arc that measures any variable angle in a circle whose radius = 1. For, since dy and ds must be homogeneous, the latter as well as the former must be a linear element. This remark applies to all the circular functions to be hereafter subjected to infinitesimal analysis.

18. Inverse Circular Functions. We have hitherto regarded the trigonometric lines as functions of arcs. If we now regard the arc as a function of one of its trigonometric lines, we shall have an *inverse circular function.* The inverse functions are designated thus,

$$\sin^{-1} y,\ \cos^{-1} y,\ \tan^{-1} y, \text{ etc.},$$

and are read respectively, *arc whose sine is* y, *arc whose cosine is* y, *arc whose tangent is* y, etc. The equations

$$s = \sin^{-1} y,\ s = \cos^{-1} y,\ s = \tan^{-1} y, \text{ etc.},$$

are nothing else than the equations

$$y = \sin s,\ y = \cos s,\ y = \tan s, \text{ etc.},$$

presented in a new form. Hence, it is from these latter that we shall derive the differentials of the former.

From $y = \sin s$ we have found $dy = \cos s\, ds$; hence

$$ds = \frac{dy}{\cos s} = \frac{dy}{\sqrt{1 - \sin^2 s}};$$

and, since $s = \sin^{-1} y$, therefore

$$d(\sin^{-1} y) = \frac{dy}{\sqrt{1 - y^2}}.$$

From $y = \cos s$ we have found $dy = -\sin s\, ds$; hence

$$ds = -\frac{dy}{\sin s} = -\frac{dy}{\sqrt{1 - \cos^2 s}};$$

and, since $s = \cos^{-1} y$, therefore

$$d(\cos^{-1} y) = -\frac{dy}{\sqrt{1-y^2}}.$$

From $y = \tan s$ we have found $dy = \frac{ds}{\cos^2 s}$; hence

$$ds = dy.\cos^2 s = \frac{dy}{1+\tan^2 s},$$

or

$$d(\tan^{-1} y) = \frac{dy}{1+y^2}.$$

From $y = \cot s$ we have found $dy = -\frac{ds}{\sin^2 s}$; hence

$$ds = -dy \sin^2 s = -\frac{dy}{1+\cot^2 s},$$

whence

$$d(\cot^{-1} y) = -\frac{dy}{1+y^2}.$$

From $y = \sec s$ we have found $dy = \frac{\sin s \, ds}{\cos^2 s}$;

hence

$$ds = dy\frac{\cos^2 s}{\sin s} = dy\frac{\cos^2 s}{\sqrt{1-\cos^2 s}} = \frac{dy}{\sec s\sqrt{\sec^2 s - 1}};$$

whence

$$d(\sec^{-1} y) = \frac{dy}{y\sqrt{y^2-1}}.$$

From $y = \operatorname{cosec} s$ we have found $dy = -\frac{\cos s \, ds}{\sin^2 s}$;

hence

$$ds = -dy\frac{\sin^2 s}{\cos s} =$$

$$-dy\frac{\sin^2 s}{\sqrt{1-\sin^2 s}} = -\frac{dy}{\operatorname{cosec} s\sqrt{\operatorname{cosec}^2 s - 1}};$$

whence

$$d(\operatorname{cosec}^{-1} y) = -\frac{dy}{y\sqrt{y^2-1}}.$$

From $y = \operatorname{vers} s$ we have found $dy = \sin s\, ds$; hence

$$ds = \frac{dy}{\sin s} = \frac{dy}{\sqrt{1-\cos^2 s}} = \frac{dy}{\sqrt{1-(1-\operatorname{vers} s)^2}},$$

which reduces to

$$d(\operatorname{vers}^{-1} y) = \frac{dy}{\sqrt{2y-y^2}}.$$

Finally, from $y = \operatorname{covers} s$ we have found $dy = -\cos s\, ds$; hence

$$ds = -\frac{dy}{\cos s} = -\frac{dy}{\sqrt{1-\sin^2 s}} =$$

$$-\frac{dy}{\sqrt{1-(1-\operatorname{covers} s)^2}},$$

which reduces to

$$d(\operatorname{covers}^{-1} y) = -\frac{dy}{\sqrt{2y-y^2}}.$$

EXAMPLES.

1. $s = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$, $\qquad ds = \frac{dx}{1+x^2}$,

2. $s = \sin^{-1}\frac{1-x^2}{1+x^2}$, $ds = -\frac{2dx}{1+x^2}$,

3. $s = \sin^{-1}(2x\sqrt{1-x^2})$, $ds = \frac{2dx}{\sqrt{1-x^2}}$,

4. $s = \cos^{-1}\frac{x}{a-x}$, $ds = -\frac{adx}{(a-x)\sqrt{a^2-2ax}}$,

5. $s = \tan^{-1}\left(\frac{a}{x}\right)$, $ds = -\frac{adx}{a^2+x^2}$,

6. $s = \text{vers}^{-1}\left(\frac{1}{x}\right)$, $ds = -\frac{dx}{x\sqrt{2x-1}}$,

7. $s = \sin^{-1}\frac{x+1}{\sqrt{2}}$, $ds = \frac{dx}{\sqrt{1-2x-x^2}}$,

8. $s = \sec^{-1}\frac{1}{2x^2-1}$, $ds = -\frac{2dx}{\sqrt{1-x^2}}$,

9. $s = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$, $ds = \frac{1}{2}dx$,

10. $s = \sin^{-1}\sqrt{\frac{a^2-x^2}{b^2-x^2}}$, $ds = -\frac{xdx\sqrt{b^2-a^2}}{(b^2-x^2)\sqrt{a^2-x^2}}$.

Functions of Two or More Variables.

19. In the preceding pages we have given the rules by which any function of one variable can be differentiated. We must now extend the same rules to the differentiation of functions of two or more variables. Let

$$u = f(x, y) \qquad (1)$$

be a function of two *independent* variables. When

x and y are made to acquire their respective increments dx and dy, the equation (1) becomes

$$u + du = f(x + dx,\ y + dy). \qquad (2)$$

Hence, subtracting (1) from (2),

$$du = f(x + dx,\ y + dy) - f(x,\ y). \qquad (3)$$

The meaning of this last equation will be better understood if we add and subtract the term $f(x + dx,\ y)$ in its second member, which we then put in the form

$$du = f(x + dx,\ y) - f(x,\ y)$$
$$+ f(x + dx,\ y + dy) - f(x + dx,\ y). \qquad (4)$$

It is obvious that the difference $f(x + dx,\ y) - f(x,\ y)$ represents the differential of the function with regard to x alone; for this difference arises exclusively from the increment dx given to x. Nor is it less obvious that the difference $f(x + dx,\ y + dy) - f(x + dx,\ y)$ represents the differential of the function with regard to y alone; for this difference arises exclusively from the increment dy given to y, the other increment dx being common to both terms, and showing that the differentiation with regard to x has already been performed.

It follows that the *total differential* of a function of two variables must consist of two parts, which are obtained by differentiating the function first with regard to x, considering y as constant, then with regard to y, considering $x + dx$ (or merely x) as constant. The total differential is represented thus,

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy, \qquad (5)$$

where $\frac{du}{dx}dx$ is the *partial differential* of the function with regard to x, and $\frac{du}{dy}dy$ its *partial differential* with regard to y. The total differential du is the sum of the partial differentials.

Had we added and subtracted the term $f(x, y+dy)$ in the second member of (3), we might have put the equation in the form

$$\begin{aligned} du = &f(x, y+dy) - f(x, y) \\ &+ f(x+dx, y+dy) - f(x, y+dy). \end{aligned} \tag{6}$$

Here the difference $f(x, y+dy) - f(x, y)$ represents the differential of the function with regard to y alone, and the other difference $f(x+dx, y+dy) - f(x, y+dy)$ represents the differential with regard to x, as obtained after the differentiation with regard to y has been performed. Now equations (4) and (6) are identical. The total differential is therefore the same, whether we differentiate the function first with regard to x, then with regard to y, or first with regard to y and then with regard to x. In other terms, *the result does not depend on the order of differentiation.*

20. We can show in the same manner that the total differential of a function of three *independent* variables is equal to the sum of the partial differentials obtained by differentiating the function with regard to each of the variables in succession. Let

$$u = \varphi(x, y, z)$$

be the given function. Its total differential will obviously be

$$du = \varphi(x + dx,\ y + dy,\ z + dz) - \varphi(x,\ y,\ z).$$

By adding and subtracting the terms $\varphi(x + dx, y, z)$ and $\varphi(x + dx, y + dy, z)$ in the second member, we may put this equation under the form

$$\begin{aligned} du = {} & \varphi(x + dx,\ y,\ z) - \varphi(x,\ y,\ z) \\ & + \varphi(x + dx,\ y + dy,\ z) - \varphi(x + dx,\ y,\ z) \\ & + \varphi(x + dx,\ y + dy,\ z + dz) - \varphi(x + dx,\ y + dy,\ z), \end{aligned}$$

where the second member consists of three differences. The first of these differences exhibits the differential of the function with respect to x alone; the second represents the differential of the function with respect to y alone; and the third represents the differential of the same function with respect to z alone. The total differential is therefore, according to our notation,

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz.$$

It would be easy to prove, in a manner quite similar to that followed in the preceding case (No. 19), that this final result is independent of the order of differentiation.

EXAMPLES.

1. $u = xy$, $\qquad du = ydx + xdy$,

2. $u = \dfrac{x}{y}$, $\qquad du = \dfrac{ydx - xdy}{y^2}$,

3. $u = x^3 - 4xy^2$, $\qquad du = (3x^2 - 4y^2)dx - 8xydy$,

4. $u = \sin(x + y)$, $\qquad du = \cos(x + y)(dx + dy)$,

5. $u = \frac{xy}{z}$, $\quad du = \frac{yzdx + xzdy - xydz}{z^2}$,

6. $u = \sqrt{x^2 + ay^2 + bz^2}$, $\quad du = \frac{xdx + aydy + bzdz}{\sqrt{x^2 + ay^2 + bz^2}}$,

7. $u = z \tan^{-1}\frac{x}{y}$, $\quad du = dz \tan^{-1}\frac{x}{y} + z\frac{ydx - xdy}{y^2 + x^2}$,

8. $u = x \log\frac{ay}{z}$, $\quad du = dx \log\frac{ay}{z} + x\frac{zdy - ydz}{zy}$.

Implicit Functions.

21. An *implicit* function is one whose value is only implicitly given in an unsolved equation. Thus $y^2 - 2xy = a^2$ is an implicit function of x; whereas, if we solve the equation, we shall have the *explicit* function

$$y = x \pm \sqrt{a^2 + x^2}.$$

When the function becomes explicit, its differential is found by the rules already given; but, as some equations cannot be readily solved, the function may remain implicit, and its differential is then to be found by the following process. Let

$$f(x, y) = 0$$

be the given function. Its differential will be

$$f(x + dx, y + dy) - f(x, y) = 0,$$

or, by adding and subtracting the term $f(x+dx, y)$,

$$f(x + dx, y) - f(x, y) + f(x + dx, y + dy) - f(x + dx, y) = 0.$$

The first of these two differences expresses the differential of the function with respect to x, and the second exhibits its differential with respect to y. Denoting the first by $\frac{df}{dx}\,dx$, and the second by $\frac{df}{dy}\,dy$, we have

$$\frac{df}{dx}dx + \frac{df}{dy}dy = 0.$$

Hence, the differential of an implicit function $f(x, y) = 0$ is obtained by differentiating it first with respect to x, as if y were constant, then with respect to y, as if x were constant, and making the sum of the results $= 0$.

Thus, from the equation $a^2y^2 + b^2x^2 - a^2b^2 = 0$, we shall obtain

$$\frac{df}{dx}dx = 2b^2xdx, \quad \frac{df}{dy}dy = 2a^2ydy,$$

and

$$2b^2xdx + 2a^2ydy = 0.$$

22. By a reasoning analogous to the above it may be shown that the differential of an implicit function of three variables, as

$$\varphi(x, y, z) = 0,$$

will be expressed by the equation

$$\frac{d\varphi}{dx}dx + \frac{d\varphi}{dy}dy + \frac{d\varphi}{dz}dz = 0.$$

SECTION II.

SUCCESSIVE DIFFERENTIALS.

23. When a function $y = f(x)$ is differentiated, its increment dy is its first differential. The differential of dy (which is written $d.dy$, or d^2y) is its second differential. The differential of d^2y (which is written $d.d^2y$, or d^3y) is its third differential; and so on.

The differential of the independent variable, inasmuch as it is the fixed standard with which the successive increments of the function are compared for determining the rate of development, is always assumed constant. Hence it does not admit of further differentiation.

If the differential of the function be divided by the differential of the variable, the quotient will be the *first differential coefficient* of the function.*

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* The authors who use *the method of limits* conceive the differential coefficient as *the limit towards which a certain variable ratio is indefinitely approaching.*

Their theory is as follows. Let a certain magnitude y depend for its value on some variable magnitude x, and suppose the relation between the two magnitudes to be expressed by the equation

$$y = x^2. \qquad (1)$$

If x takes the increment h, and if the corresponding value of y be represented by y', we shall have

$$y' = (x + h)^2 = x^2 + 2xh + h^2, \qquad (2)$$

and subtracting (1) from (2),

$$y' - y = 2xh + h^2,$$

whence

$$\frac{y' - y}{h} = 2x + h. \qquad (3)$$

The differential coefficient of the first differential coefficient will be the *second differential coefficient* of the function: the differential coefficient of the

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In this last equation, the term $2x$ being independent of h, this increment may undergo any change of value without affecting $2x$. Let, then, h continually decrease till it becomes $= 0$. The expression for the ratio $\frac{y'-y}{h}$ will then be simply $2x$. Hence $2x$ is *the limit* toward which the ratio $\frac{y'-y}{h}$ approaches as h is diminished: which limit the ratio cannot reach until h becomes zero. Such is the process by which differential coefficients are determined in the theory or method of limits.

This theory, though still fashionable in France and elsewhere, labors under great radical defects. We remark, first, that when $h = 0$, then also $y' - y = 0$. Hence, at the limit, the first member of the equation (3), though represented by $\frac{dy}{dx}$ in order to keep a trace of the variables, would really be $\frac{0}{0}$. Now, to assume that h, dx, and dy are absolute zeros is to assume that the limit has been reached, whereas the theory itself teaches that $h = 0$ can never be reached, inasmuch as the hypothesis $h = 0$ would exclude all idea of change or continuity. If, then, the assumption $h = 0$ can never be true, how can we assume $h = 0$, and accept equations based on such an assumption?

We must also remember that from $\frac{dy}{dx} = 2x$ we derive

$$y = \int 2xdx,$$

in which expression, if dx were an absolute zero, what would y be but a mere sum of nothings?

Again, the symbol $\frac{dy}{dx}$ represents, as is well-known, the trigonometric tangent of the angle that an element of the curve makes with the axis of abscissas, the element itself being represented by $ds = \sqrt{dx^2 + dy^2}$. Now, this element is not an absolute zero; for the absolute zero, or a mere point at rest, cannot form any particular angle with the axis. Hence ds must be a real quantity; and, if so, dx and dy are also real quantities. Accordingly the theory which considers them as limits of decreasing quantities is not consistent with itself.

Prof. Todhunter, a follower of this theory, to eschew objections, declares that *he considers the symbol* $\frac{\mathrm{dy}}{\mathrm{dx}}$ *as a whole, and does not assign a separate meaning to* dy *and* dx, though *he knows that the student will very possibly* (and very reasonably, too) *suspect that some meaning may be given to* dx *and* dy *which will enable him to regard* $\frac{\mathrm{dy}}{\mathrm{dx}}$ *as a fraction.* The student, however, might humbly remind the Professor that the ratio $\frac{dy}{dx}$ represents only a particular state of the ratio $\frac{y'-y}{h}$, and that, as this latter, so also the former is a result of division; and, therefore, that dy is a real numerator, and dx a real denominator. And as to the separate meanings of dy and dx, it is not difficult to see that, if x and y be considered as co-ordinates of a point in motion, dx and dy will represent the developments which x and y are ac-second differential coefficient is the *third differential coefficient* of the function; and so on.

For example, let $y = ax^4$. The successive differentials of this function will be the following,

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quiring in a given infinitesimal instant dt: and if v and v' be the velocities with which they develop in that instant, then $dx = vdt$ and $dy = v'dt$; whence

$$\frac{dy}{dx} = \frac{v'dt}{vdt} = \frac{v'}{v}.$$

Thus it is plain that dy and dx have their separate meaning, and that the ratio $\frac{dy}{dx}$ is the ratio of the velocities with which x and y are developing at a given instant. It is evident, therefore, that dy and dx are *real* and *distinct* quantities, which correspond to h infinitesimal, and not to the limit $h = 0$.

Perhaps it will be said that, in the theory of limits, *the words* "when $h = 0$" *are only an abbreviation for the words* "when h is continually diminishing towards zero." This is, indeed, what Professor Todhunter explicitly teaches (*Diff. Calc.* § 9). But it is obvious that, if h is only diminishing towards zero and never reaches the limit zero, the theory of limits remains without object, and virtually abdicates in favor of the old doctrine of infinitesimals. For *infinitesimals*, as defined by the advocates of the theory of limits (Duhamel, *Diff. Calc.*), are just *such variable magnitudes as tend to the limit zero*, though they never reach such a limit.

To this doctrine of limits we must object another serious defect regarding its method of working. The theory *needlessly* starts by giving to the independent variable x the finite increment h, thus creating for itself the strange necessity of running for ever after the limit $h = 0$, which limit the theory declares to be unattainable. This is against nature and against reason. The variable x, when continuously increasing, does not change suddenly into $x + h$, but it changes directly into $x + dx$. What is then the use of travelling the whole distance h, if it is necessary to travel it back again? Is it natural, when dx presents itself and is within immediate reach, to wander away from it, with no hope of finding it again, except *perhaps* after a long and circuitous journey? Is it not more natural, and therefore more reasonable, to pass from x to $x + dx$ directly? The infinitesimal increment dx precedes, in the order of causality, the finite increment h; hence to derive dx from h, and to say that dx is the limit of a continually decreasing quantity, is to overturn the order of causality, just as if we said that the acorn is the limit of a continually decreasing oak, or that the most rudimentary human embryo is the limit of a continually decreasing baby.

It will be said that all writers on differential calculus, whatever be their theory, always begin with $x + h$, and assume h to be a finite quantity.—Yes; all authors, *in making their diagrams*, give to h a finite value, because infinitesimals cannot be represented or shown in a drawing; but, though they mark out a finite increment in order to make it visible to the student, they do not mean that the increment is finite: they simply mean that there is an increment, and they declare that what appears as finite in the diagram must be considered as a mere infinitesimal, or a nascent quantity. Hence they have no need of going to and fro in search of limit unattainable, but they immediately take hold of the nascent quantity dx,

$$dy = 3ax^2dx,\ d^2y = 6axdx^2,\ d^3y = 6adx^3,\ d^4y = 0,$$

and the successive differential coefficients will be

$$\frac{dy}{dx} = 3x^2, \frac{d^2y}{dx^2} = 6x\ \frac{d^3y}{dx^3} = 6, \frac{d^4y}{dx^4} = 0.$$

EXAMPLES.

1. $y = ax^n, \dfrac{dy}{dx} = nax^{n-1}, \dfrac{d^2y}{dx^2} = n(n-1)ax^{n-2},$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)\ ax^{n-3}, \ldots$$

2. $y = \log(x+1), \dfrac{dy}{dx} = \dfrac{1}{x+1}, \dfrac{d^2y}{dx^2} = -\dfrac{1}{(x+1)^2},$

$$\frac{d^3y}{dx^3} = \frac{1.2}{(x+1)^3}, \frac{d^4y}{dx^4} = -\frac{1.2.3}{(x+1)^4}, \ldots$$

3. $y = \sin x, \dfrac{dy}{dx} = \cos x, \dfrac{d^2y}{dx^2} = -\sin x,$

$$\frac{d^3y}{dx^3} = -\cos x, \frac{d^4y}{dx^4} = \sin x, \ldots$$

4. $y = a^x, \dfrac{dy}{dx} = a^x \log a, \dfrac{d^2y}{dx^2} = a^x(\log a)^2,$

$$\frac{d^3y}{dx^3} = a^x(\log a)^3, \frac{d^4y}{dx^4} = a^x(\log a)^4, \ldots$$

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which is, in fact, the immediate increment of the variable x, and which is the only quantity required by the process of differentiation.

From these remarks, and from a right conception of the nature of infinitesimals as laid down in our Introduction, the reader will understand how we were logically compelled to adopt the infinitesimal method: and we are confident that even the advocates of the theory of limits, if they examine the subject in a truly philosophical spirit, will not be loath to recognize that our doctrine enjoys the great relative advantage of being able to account for itself without any of those inconsistencies that mar the substance, or at least the exposition, of their favorite theory.

5. $y = xe^x$, $\frac{dy}{dx} = (x+1)\,e^x$, $\frac{d^2y}{dx^2} = (x+2)\,e^x$,

$$\frac{d^3y}{dx^3} = (x+3)\,e^x, \ \ldots \ \frac{d^ny}{dx^n} = (x+n)\,e^x.$$

Maclaurin's Formula.

24. By the aid of successive differentials many functions have been developed into series, from which the functions themselves can be easily calculated. The formulas most used for such developments are Maclaurin's and Taylor's formulas.

By Maclaurin's formula the function of one variable is developed into a series arranged according to the ascending powers of that variable, the coefficients being constant. Thus, if y is a function of x, and if it admits of being developed in the form

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots \quad (1)$$

the constants A, B, C, D, . . . will be determined by successive differentiations. Differentiating (1), and dividing by dx, we find

$$\frac{dy}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots \quad (2)$$

Differentiating (2), and dividing by dx, we find

$$\frac{d^2y}{dx^2} = 2C + 2.3Dx + 3.4\,Ex^2 + \ldots \quad (3)$$

Differentiating (3), and dividing by dx, we find

$$\frac{d^3y}{dx^3} = 2.3D + 2.3.4Ex + \ldots \quad (4)$$

and so on. Now, as x may have any value consistent with the convergency of the series, assume $x=0$, and let

$$(y),\ \left(\frac{dy}{dx}\right),\ \left(\frac{d^2y}{dx^2}\right),\ \ldots$$

be what y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, . . . become in this hypothesis. Then

$$(y)=A,\ \left(\frac{dy}{dx}\right)=B,\ \left(\frac{d^2y}{dx^2}\right)=2C,\ \left(\frac{d^3y}{dx^3}\right)=2.3D,\ \ldots$$

whence

$$A=(y), B=\left(\frac{dy}{dx}\right), C=\frac{1}{2}\left(\frac{d^2y}{dx^2}\right), D=\frac{1}{2.3}\left(\frac{d^3y}{dx^3}\right),\ \ldots$$

and, substituting these values in (1),

$$y=(y)+\left(\frac{dy}{dx}\right)x+\left(\frac{d^2y}{dx^2}\right)\frac{x^2}{2}+\left(\frac{d^3y}{dx^3}\right)\frac{x^3}{2.3} + \left(\frac{d^4y}{dx^4}\right)\frac{x^4}{2.3.4}+\ \ldots \quad (5)$$

This is Maclaurin's formula. In using it, it is necessary, of course, that the values attributed to x be such as will make the series *convergent.*

If a function is not susceptible of development by this formula, the formula itself will give notice of the fact; for, in such a case, some of its constant factors will become infinite.

EXAMPLES.

1. To develop $y=\sin x$. We have

$$y=\sin x,\ \frac{dy}{dx}=\cos x, \frac{d^2y}{dx^2}=-\sin x, \frac{d^3y}{dx^3}=-\cos x,\ \ldots$$

hence

$$(y)=0,\ \left(\frac{dy}{dx}\right)=1,\ \left(\frac{d^2y}{dx^2}\right)=0,\ \left(\frac{d^3y}{dx^3}\right)=-1,\ \ldots$$

and substituting in the formula (5),

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \ldots$$

This series, being differentiated and then divided by dx, gives

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \ldots$$

2. To develop $y=(1+x)^n$. We have

$$y=(1+x)^n,\ \frac{dy}{dx}=n(1+x)^{n-1},$$
$$\frac{d^2y}{dx^2}=n(n-1)(1+x)^{n-2},$$
$$\frac{d^3y}{dx^3}=n(n-1)(n-2)(1+x)^{n-3},\ \ldots$$

hence

$$(y)=1,\ \left(\frac{dy}{dx}\right)=n,\ \left(\frac{d^2y}{dx^2}\right)=n(n-1),$$
$$\left(\frac{d^3y}{dx^3}\right)=n(n-1)(n-2),\ \ldots$$

and therefore

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{2.3}x^3 + \ldots$$

3. To develop $y=\log(1+x)$. We have

$$y=\log(1+x),\ \frac{dy}{dx}=\frac{1}{1+x},$$
$$\frac{d^2y}{dx^2}=-\frac{1}{(1+x)^2},\ \frac{d^3y}{dx^3}=\frac{1.2}{(1+x)^3},\ \ldots$$

hence

$$(y) = 0, \left(\frac{dy}{dx}\right) = 1, \left(\frac{d^2y}{dx^2}\right) = -1, \left(\frac{d^3y}{dx^3}\right) = 2, \ldots$$

and therefore

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

4. To develop $y = a^x$. We have

$$y = a^x, \frac{dy}{dx} = a^x \log a, \frac{d^2y}{dx^2} = a^x (\log a)^2,$$

$$\frac{d^3y}{dx^3} = a^x (\log a)^3, \ldots$$

whence

$$(y) = 1, \left(\frac{dy}{dx}\right) = \log a, \left(\frac{d^2y}{dx^2}\right) = (\log a)^2,$$

$$\left(\frac{d^3y}{dx^3}\right) = (\log a)^3, \ldots$$

and therefore

$$a^x = 1 + (\log a)x + \frac{(\log a)^2}{2} x^2 + \frac{(\log a)^3}{2.3} x^3 + \ldots$$

If we make $a = e$, whence $\log a = \log e = 1$, then we have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \ldots$$

Taylor's Formula.

25. Let $u = f(x)$ and $u' = f(x + h)$. Considering x and h as two arbitrary parts of a certain line, it is obvious that, if the line receives an infinitesimal increment, the result will be the same, whether the increment be attached to the part x or to the part h. In other terms, the result will be the same whether the function $u' = f(x + h)$ be differentiated with regard to x, considering h as constant, or with regard to h, considering x as constant. In the first case, the differential coefficient of the function will be $\frac{du'}{dx}$: in the second case it will be $\frac{du'}{dh}$; and therefore we shall have

$$\frac{du'}{dx} = \frac{du'}{dh} \quad . \quad . \quad . \quad . \qquad (1)$$

This equation will afford us the means of developing the function $u' = f(x+h)$ into a series arranged according to the ascending powers of h, with coefficients that are functions of x alone.

Let us assume a development of the form

$$u' = P + Qh + Rh^2 + Sh^3 + Th^4 + \ldots \qquad (2)$$

in which P, Q, R, . . . are functions of x alone. Differentiating (2) with regard to x, and dividing by dx, we shall find

$$\frac{du'}{dx} = \frac{dP}{dx} + \frac{dQ}{dx}h + \frac{dR}{dx}h^2 + \frac{dS}{dx}h^3 + \frac{dT}{dx}h^4 + \ldots \qquad (3)$$

then, differentiating (2) with regard to h, and dividing by dh,

$$\frac{du'}{dh} = Q + 2Rh + 3Sh^2 + 4Th^3 + \ldots \qquad (4)$$

Now, by (1), the first members of (3) and (4) are equal; hence their second members are also equal, and the coefficients of like powers of h in those second members are equal. Therefore

$$\frac{dP}{dx} = Q, \frac{dQ}{dx} = 2R, \frac{dR}{dx} = 3S, \frac{dS}{dx} = 4T \ldots$$

But P is the value u of the function when $h=0$; and therefore $dP=du$; hence we shall have

$$P=u,\ Q=\frac{du}{dx},\ R=\frac{1}{2}\cdot\frac{d^2u}{dx^2},\ S=\frac{1}{2.3}\cdot\frac{d^3u}{dx^3},$$
$$T=\frac{1}{2.3.4}\cdot\frac{d^4u}{dx^4},\ \ldots$$

These values substituted in the equation (2) give

$$u'=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\cdot\frac{h^2}{2}+\frac{d^3u}{dx^3}\cdot\frac{h^3}{2.3}$$
$$+\frac{d^4u}{dx^4}\cdot\frac{h^4}{2.3.4}+\ldots \quad (5)$$

This is Taylor's formula. The values attributed to x and h must be such as will render the series *convergent.*

EXAMPLES.

1. Let $u=x^n$; then $u'=(x+h)^n$; and we have

$$\frac{du}{dx}=nx^{n-1},\ \frac{d^2u}{dx^2}=n(n-1)x^{n-2},$$
$$\frac{d^3u}{dx^3}=n(n-1)(n-2)x^{n-3},\ \ldots$$

hence

$$u'=(x+h)^n=x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2$$
$$+\frac{n(n-1)(n-2)}{2.3}x^{n-3}h^3+\ldots$$

2. Let $u=\log x$; then $u'=\log(x+h)$; and we have

$$\frac{du}{dx}=\frac{1}{x},\ \frac{d^2u}{dx^2}=-\frac{1}{x^2},\ \frac{d^3u}{dx^3}=\frac{2}{x^3},\ \frac{d^4u}{dx^4}=-\frac{2.3}{x^4},\ \ldots$$

whence

$$u' = \log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \ldots$$

3. Let $u = e^x$; then $u' = e^{x+h}$; and we have

$$\frac{du}{dx} = e^x, \; \frac{d^2u}{dx^2} = e^x, \; \frac{d^3u}{dx^3} = e^x, \ldots$$

whence

$$u' = e^{x+h} = e^x \left(1 + h + \frac{h^2}{2} + \frac{h^3}{2.3} + \frac{h^4}{2.3.4} + \ldots\right).$$

In this equation make $x = 0$ and $h = 1$. Then we have

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \ldots$$

or

$$e = 2.718281828459 \ldots,$$

which is the basis of the hyperbolic logarithms.

4. Let $u = \sin x$, and $u' = \sin(x+y)$; then we have

$$\frac{du}{dx} = \cos x, \; \frac{d^2u}{dx^2} = -\sin x, \; \frac{d^3u}{dx^3} = -\cos x, \ldots$$

whence

$$u' = \sin(x+y) = \sin x + y\cos x - \frac{y^2}{2}\sin x$$
$$- \frac{y^3}{2.3}\cos x + \frac{y^4}{2.3.4}\sin x + \frac{y^5}{2.3.4.5}\cos x - \ldots$$

If we change y into $-y$, we shall have also

$$\sin(x-y) = \sin x - y\cos x - \frac{y^2}{2}\sin x$$
$$+ \frac{y^3}{2.3}\cos x + \frac{y^4}{2.3.4}\sin x + \ldots$$

SCHOLIUM. Taylor's formula may be used for the development of a function $u = f(x, y)$ of two independent variables.

If we begin by giving to x an increment h, we shall have

$$f(x+h, y) = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{2} + \frac{d^3u}{dx^3}\frac{h^3}{2.3} + \ldots$$

and in this equation, when we give to y an increment k, the first member will become $f(x+h, y+k)$, and in the second member u will become

$$u + \frac{du}{dy}k + \frac{d^2u}{dy^2}\frac{k^2}{2} + \frac{d^3u}{dy^3}\frac{k^3}{2.3} + \ldots$$

$\frac{du}{dx}h$ will become

$$\left(\frac{du}{dx} + \frac{d\left(\frac{du}{dx}\right)}{dy}k + \frac{d^2\left(\frac{du}{dx}\right)}{dy^2}\frac{k^2}{2} + \ldots\right)h,$$

$\frac{d^2u}{dx^2}\frac{h^2}{2}$ will become

$$\left(\frac{d^2u}{dx^2} + \frac{d\left(\frac{d^2u}{dx^2}\right)}{dy}k + \frac{d^2\left(\frac{d^2u}{dx^2}\right)}{dy^2}\frac{k^2}{2} + \ldots\right)\frac{h^2}{2},$$

and so on.

Substituting these values in the above equation, we shall have

$$\begin{aligned} f(x+h, y+k) = u &+ \frac{du}{dy}k + \frac{d^2u}{dy^2}\frac{k^2}{2} + \frac{d^3u}{dy^3}\frac{k^3}{2.3} + \ldots \\ &+ \frac{du}{dx}h + \frac{d\left(\frac{du}{dx}\right)}{dy}hk + \frac{d^2\left(\frac{du}{dx}\right)}{dy^2}\frac{hk^2}{2} + \ldots \\ &+ \frac{d^2u}{dx^2}\frac{h^2}{2} + \frac{d\left(\frac{d^2u}{dx^2}\right)}{dy}\frac{h^2k}{2} + \ldots \\ &+ \ldots \end{aligned} \qquad \text{(A)}$$

We might have begun by giving to y the increment k; and in this case we would have directly obtained

$$f(x, y+k) = u + \frac{du}{dy}k + \frac{d^2u}{dy^2}\frac{k^2}{2} + \frac{d^3u}{dy^3}\frac{k^3}{2.3} + \ldots$$

In this equation, when x receives the increment h, the first member becomes $f(x+h,\ y+k)$, as before, and in the second member u becomes

$$u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{2}+\frac{d^3u}{dx^3}\frac{h^3}{2.3}+\ldots$$

$\frac{du}{dy}k$ becomes

$$\left(\frac{du}{dy}+\frac{d\left(\frac{du}{dy}\right)}{dx}h+\frac{d^2\left(\frac{du}{dy}\right)}{dx^2}\frac{h^2}{2}+\ldots\right)k,$$

$\frac{d^2u}{dy^2}\frac{k^2}{2}$ becomes

$$\left(\frac{d^2u}{dy^2}+\frac{d\left(\frac{d^2u}{dy^2}\right)}{dx}h+\frac{d^2\left(\frac{d^2u}{dy^2}\right)}{dx^2}\frac{h^2}{2}+\ldots\right)\frac{k^2}{2},$$

and so on.

Substituting these values in the above equation, we shall have

$$\begin{aligned}f(x+h,\ y+k)=u&+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{2}+\frac{d^3u}{dx^3}\frac{h^3}{2.3}+\ldots\\&+\frac{du}{dy}k+\frac{d\left(\frac{du}{dy}\right)}{dx}hk+\frac{d^2\left(\frac{du}{dy}\right)}{dx^2}\frac{h^2k}{2}+\ldots\\&+\frac{d^2u}{dy^2}\frac{k^2}{2}+\frac{d\left(\frac{d^2u}{dy^2}\right)}{dx}\frac{hk^2}{2}+\ldots\\&+\ldots\end{aligned}\qquad\text{(B)}$$

Remembering that $u=f(x,\ y)$, the equations (A) and (B) may be written as follows:

$$f(x+h,\ y+k)-f(x,\ y)=\frac{du}{dy}k+\frac{du}{dx}h+\frac{d\left(\frac{du}{dx}\right)}{dy}hk+\ldots$$

$$f(x+h,\ y+k)-f(x,\ y)=\frac{du}{dx}h+\frac{du}{dy}k+\frac{\left(\frac{du}{dy}\right)}{dx}hk+\ldots$$

and, as (A) and (B) are identical, we conclude that the two coefficients of hk are identical; whence

$$\frac{d\left(\frac{du}{dx}\right)}{dy} = \frac{d\left(\frac{du}{dy}\right)}{dx}. \qquad \text{(C).}$$

When h and k are infinitesimal, the equations (A) and (B) reduce to

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy,$$

which is the total differential of the function (No. 19). It is, therefore, the property of such a total differential that the differential coefficient of $\frac{du}{dx}$ taken with respect to y is equal to the differential coefficient of $\frac{du}{dy}$ taken with respect to x.

De Moivre's Formulas.

26. We have found (No. 24) the three series

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \ldots \qquad (1)$$

$$\sin x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + \ldots \qquad (2)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \ldots \qquad (3)$$

If we change x into $x\sqrt{-1}$ in this last series, we shall have

$$e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{2} - \frac{x^3\sqrt{-1}}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5\sqrt{-1}}{2.3.4.5} - \ldots \qquad (4)$$

If, on the other hand, we multiply both members of (2) by $\sqrt{-1}$, and then add the result to (1), we shall find that the sum

$$\cos x + \sqrt{-1}\sin x$$

will give a series identical with that of equation (4). Therefore

$$\cos x + \sqrt{-1}\sin x = e^{x\sqrt{-1}}. \qquad (5)$$

If we change x into $-x$, we shall have also

$$\cos x - \sqrt{-1}\sin x = e^{-x\sqrt{-1}}. \qquad (6)$$

And now let $x = mz$, m being any number; then

$$\cos mz + \sqrt{-1}\sin mz = e^{mz\sqrt{-1}} = (e^{z\sqrt{-1}})^m,$$

that is,

$$\cos mz + \sqrt{-1}\sin mz = (\cos z + \sqrt{-1}\sin z)^m, \quad (7)$$

as also

$$\cos mz - \sqrt{-1}\sin mz = (\cos z - \sqrt{-1}\sin z)^m. \quad (8)$$

These are De Moivre's formulas. Each of them is equivalent to two; for, after developing the second members, the *real* terms will form an equation among themselves, and the *imaginary* terms will form another, which being divided by the common factor $\sqrt{-1}$, will express another real relation. Thus, making $m = 3$, formula (7) will give

$$\cos 3z = \cos^3 z - 3\cos z\sin^2 z,$$

$$\sin 3z = 3\cos^2 z\sin z - \sin^3 z,$$

Maxima and Minima.

27. A function is a *maximum* when it reaches a value greater than the values immediately preceding and immediately following, and it is a *minimum* when it reaches a value less than the values immediately preceding and immediately following.

To determine if a function $u = f(x)$ has a maximum or a minimum for a certain value of x, the following plain rules are laid down.

If the function, for a certain value of x, becomes a maximum, it must reach that maximum *by a last increase*, and then begin to diminish. Hence the differential du must be *positive* immediately before, and *negative* immediately after the maximum.

If the function, for a certain value of x, becomes a minimum, it must reach that minimum *by a last diminution*, and then begin to increase. Hence the differential du must be *negative* immediately before, and *positive* immediately after the minimum.

When du changes from $+$ to $-$ or from $-$ to $+$, so also does the differential coefficient $\frac{du}{dx}$. And, as a quantity subject to the law of continuity can change its sign only by becoming zero, or infinity, no value of the variable will give a maximum or a minimum value to the function, unless it reduces $\frac{du}{dx}$ to zero or to infinity. Hence the roots of the equations

$$\frac{du}{dx} = 0, \text{ and } \frac{du}{dx} = \infty,$$

will give all the values of x which can possibly make u a maximum or a minimum.

Let a be one of such roots. It is yet necessary to ascertain whether $x = a$ corresponds to a maximum or to a minimum. If in the expression of $\frac{du}{dx}$ we put first $a - dx$, then $a + dx$, instead of x, and if the first result be *positive*, and the second *negative*, it is plain, from the preceding considerations, that the value $x = a$ will correspond to a maximum; if the first result be *negative* and the second *positive*, the value $x = a$ will correspond to a minimum; but if both results be of the same sign, there will be no maximum and no minimum for $x = a$.

Assume, as a first example, the function

$$y = (a - x)^2 + b,$$

which gives

$$\frac{dy}{dx} = -2(a - x).$$

Making $-2(a - x) = 0$, we find $x = a$. Putting $a - dx$, then $a + dx$, instead of x, the expression $-2(a - x)$ becomes successively

$$-2(a - (a - dx)) = -2dx,$$
$$-2(a - (a + dx)) = +2dx;$$

and, since the first result is negative and the second positive, we see that $x = a$ corresponds to a minimum, which is $y = b$.

The hypothesis $-2(a - x) = \infty$ gives $x = \infty$, which cannot verify the conditions of either a maximum or a minimum.

Assume, as a second example, the function

$$y^2 = 2ax - x^2,$$

from which we have

$$\frac{dy}{dx} = \pm \frac{a - x}{\sqrt{2ax - x^2}}.$$

Making $\pm(a - x) = 0$, we have $x = a$. Putting

$a - dx$, and then $a + dx$, instead of x, the expression $+(a - x)$ becomes

$$+a - (a - dx) = + dx, \qquad + a - (a + dx) = - dx,$$

whilst the expression $-(a - x)$ becomes

$$-(a - (a - dx)) = - dx, \; -(a - (a + dx)) = + dx.$$

Hence $x = a$ corresponds to a maximum and to a minimum at the same time, owing to the double sign of $\frac{dy}{dx}$. The maximum is $y = a$, and the minimum $y = -a$.

The hypothesis $\frac{a - x}{\sqrt{2ax - x^2}} = \infty$ gives $2ax - x^2 = 0$, whence $x = 0$, or $x = 2a$. But in both cases y becomes $= 0$; and thus there is no other maximum or minimum.

28. When a function of x can be developed by Taylor's formula, the determination of its maxima and minima can be made to depend entirely on its successive differential coefficients.

Let the function $u = f(x)$ be at its maximum or minimum, and let

$$u' = f(x - h) \text{ and } u'' = f(x + h)$$

be the values of u immediately before and immediately after the maximum or minimum. By Taylor's formula we have

$$u' = u - \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{2} - \frac{d^3u}{dx^3}\frac{h^3}{2.3} + \dots$$

$$u'' = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{2} + \frac{d^3u}{dx^3}\frac{h^3}{2.3} + \dots$$

whence

$$u' - u = h\left(-\frac{du}{dx} + \frac{d^2u}{dx^2}\frac{h}{2} - \frac{d^3u}{dx^3}\frac{h^2}{2.3} + \cdots\right)$$

$$u'' - u = h\left(+\frac{du}{dx} + \frac{d^2u}{dx^2}\frac{h}{2} + \frac{d^3u}{dx^3}\frac{h^2}{2.3} + \cdots\right)$$

If u is a maximum, then $u > u'$, and $u > u''$; hence the two series must be both negative. If u is a minimum, then $u < u'$, and $u < u''$, and the two series must be both positive. Now, the two series cannot have equal signs unless the terms $-\frac{du}{dx}$ and $+\frac{du}{dx}$ disappear; for, when h is very small (as it must be in the present case), the sign of each series is the same as that of its first term. Hence we must have

$$\frac{du}{dx} = 0,$$

and the roots of this equation will give the values of x for which the function u will be a maximum or a minimum.

Let a be one of these roots. As $\left(\frac{du}{dx}\right)_a$ disappears, the two series reduce to

$$(u' - u)_a = \left(\frac{d^2u}{dx^2}\right)_a \frac{h^2}{2} - \left(\frac{d^3u}{dx^3}\right)_a \frac{h^3}{2.3} + \cdots,$$

$$(u'' - u)_a = \left(\frac{d^2u}{dx^2}\right)_a \frac{h^2}{2} + \left(\frac{d^3u}{dx^3}\right)_a \frac{h^3}{2.3} + \cdots,$$

and their signs will be those of $\left(\frac{d^2u}{dx^2}\right)_a$. If this term be negative, u_a will be a maximum: if it be positive, u_a will be a minimum.

If $x = a$ were to give also $\left(\frac{d^2u}{dx^2}\right)_a = 0$, then the two series would reduce to

$$u' - u = -\left(\frac{d^3u}{dx^3}\right)_a \frac{h^3}{2.3} + \left(\frac{d^4u}{dx^4}\right)_a \frac{h^4}{2.3.4} - \dots,$$

$$u'' - u = +\left(\frac{d^3u}{dx^3}\right)_a \frac{h^3}{2.3} + \left(\frac{d^4u}{dx^4}\right)_a \frac{h^4}{2.3.4} + \dots,$$

and thus they would again have different signs. Hence there could be no maximum and no minimum, unless $\left(\frac{d^3u}{dx^3}\right)_a = 0$. In this case the two series would begin by the term $\left(\frac{d^4u}{dx^4}\right)_a \frac{h^4}{2.3.4}$, which, if positive, would give a minimum, and if negative, a maximum. If this term also were to become $= 0$, we would have to proceed as before with regard to the subsequent terms of the series. From all this we may draw the following conclusion:

If the first differential coefficient which does not become $= 0$ is of an uneven order, the two series have opposite signs, and there is no maximum or minimum. If the first differential coefficient which does not become $= 0$ is of an even order, the two series have equal signs, and there will be a maximum when their sign is negative, and a minimum when their sign is positive.

29. The investigation of maxima and minima may often be simplified. Thus *a constant factor* *that affects the whole function can be suppressed* in the differentiation; for, since we have $\frac{du}{dx}=0$, the result is independent of any such factor.

So also, if we have a function of the form $y=\sqrt{a^2x-bx^2}$, we can make $y^2=u=a^2x-bx^2$, whence $\frac{du}{dx}=a^2-2bx$, and $\frac{d^2u}{dx^2}=-2b$. Making $\frac{du}{dx}=0$, we have $x=\frac{a^2}{2b}$. This value makes u a maximum; hence it makes y^2 a maximum, and consequently also y a maximum. By this artifice *we can differentiate the function without taking notice of the radical sign.*

And, again, we may simplify operations *by taking the logarithms of the quantities to be differentiated.* Thus, if we have a function

$$y=\frac{(x-1)\,(x-2)}{(x+1)\,(x+2)},$$

passing to logarithms, and making $\log y=u$, we shall have

$$u=\log(x-1)+\log(x-2)-\log(x+1)-\log(x+2),$$

and

$$\frac{du}{dx}=\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x+1}-\frac{1}{x+2}=0;$$

from which we obtain $x=\pm\sqrt{2}$. It is obvious that, when this value of x makes u a maximum or a minimum, y also will be a maximum or a minimum.

When the first differential coefficient is a product of two or more factors, and one of these factors becomes $=0$ for a value of x corresponding to a maximum or to a minimum, *the second differential coefficient can be obtained without differentiating the other factors*, as in the following example. Let

$$\frac{du}{dx} = P \times Q \times R,$$

P, Q, and R being functions of x. The regular differentiation would give

$$\frac{d^2u}{dx^2} = QR\frac{dP}{dx} + PR\frac{dQ}{dx} + PQ\frac{dR}{dx}.$$

Now, if the factor R, for instance, becomes $= 0$ for a value $x = a$, it is evident that the differential will reduce simply to

$$\left(\frac{d^2u}{dx^2}\right)_a = \left(PQ\frac{dR}{dx}\right)_a.$$

Hence it will suffice, in such a case, to multiply the other factors by the differential coefficient of the factor which becomes $= 0$.

Exercises on Maxima and Minima.

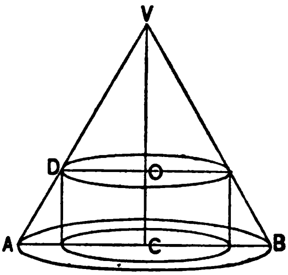
30. The application of the preceding principles to the solution of problems is not difficult, though the student may, at times, experience some difficulty in finding out the mode of expressing the particular function which is to be worked upon. A few examples will show how the difficulty may be practically overcome.

I. *Required the dimensions of the maximum cylinder that can be inscribed in a given right cone.*

Let AVB (Fig. 1) be the cone, and suppose a cylinder inscribed. Let $VC = h$, $AC = r$, $VO = y$, $DO = x$. The volume V of the cylinder will be expressed by

$$V = \pi x^2 (h - y).$$

Fig. 1



But from the similar triangles AVC and DVO we have

$x : y :: AC : VC :: r : h$, and

$$y = \frac{hx}{r}.$$

Substituting this value of y in the preceding expression, we have

$$V = \frac{\pi h}{r} x^2 (r - x).$$

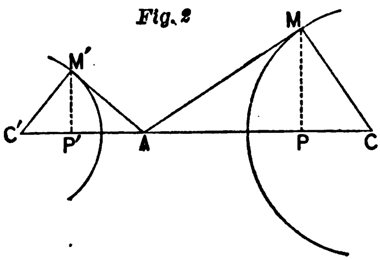
Hence

$$\frac{dV}{dx} = \frac{\pi h}{r}(2rx - 3x^2), \quad \frac{d^2V}{dx^2} = \frac{\pi h}{r}(2r - 6x).$$

Placing $\frac{dV}{dx} = 0$, we find the roots $x = 0$ and $x = \frac{2r}{3}$. The first value placed in the expression of $\frac{d^2V}{dx^2}$ makes it positive, the second makes it negative. Hence $x = 0$ corresponds to a minimum, and $x = \frac{2r}{3}$ to the maximum required, which will be $V = \frac{4\pi hr^2}{27}$. Its altitude is $= \frac{h}{3}$, whilst the radius of its base is $= \frac{2r}{3}$.

II. *In the line CC'* (Fig. 2) *which joins the centres of two spheres, to find the point from which the greatest portion of spherical surface is visible.*

Fig. 2



Let A be the point sought for. Draw the tangents AM and AM', the radii $CM = r$, $C'M' = r'$, and the lines MP, $M'P'$ perpendicular to CC'. Make $CC' = a$, and $AC = x$.

The portion of surface visible from A on the right hand is one-half of the sphere, minus a zone of the altitude PC, or

$$2\pi r^2 - 2\pi r \times PC;$$

and the portion visible on the left hand is one-half of the surface of the other sphere, minus a zone of the altitude $P'C'$, or

$$2\pi r'^2 - 2\pi r' \times P'C'.$$

Calling s the total visible surface, we have

$$s = 2\pi (r^2 + r'^2 - r \times PC - r' \times P'C').$$

But

$$PC : r :: r : x, \text{ and } P'C' : r' :: r' : a - x;$$

hence

$$PC = \frac{r^2}{x}, \text{ and } P'C' = \frac{r'^2}{a - x}; \text{ and therefore}$$

$$s = 2\pi \left(r^2 + r'^2 - \frac{r^2}{x} - \frac{r'^2}{a - x}\right);$$

whence

$$\frac{ds}{dx} = 2\pi\left(\frac{r^3}{x^2} - \frac{r'^3}{(a-x)^2}\right);$$

and making $\frac{ds}{dx} = 0$, we shall find

$$x = a\,\frac{\sqrt{r^3}}{\sqrt{r^3} + \sqrt{r'^3}}\,.$$

Such is, then, the distance from the centre C to the point A that satisfies the condition expressed in the enunciation of the problem.

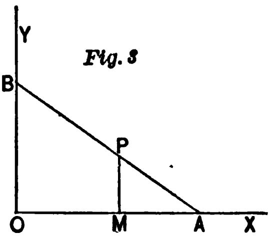


Fig. 3

III. *Through a point P* (Fig. 3) *a straight line is drawn meeting the axes OX and OY at A and B respectively. Find the least length that this line can have.*

Let $OM = a$, and $PM = b$ be the co-ordinates of the given point P, and make the angle $PAO = \vartheta$. Then

$$PA = \frac{b}{\sin\vartheta}, \text{ and } PB = \frac{a}{\cos\vartheta};$$

hence, making $AB = u$, we shall have

$$u = \frac{b}{\sin\vartheta} + \frac{a}{\cos\vartheta};$$

whence

$$\frac{du}{d\vartheta} = -\frac{b\cos\vartheta}{\sin^2\vartheta} + \frac{a\sin\vartheta}{\cos^2\vartheta}.$$

Making $\frac{du}{d\vartheta} = 0$ we shall find

$$\tan\vartheta = \left(\frac{b}{a}\right)^{\frac{1}{3}}.$$

From this last equation we obtain

$$\frac{a}{\cos\vartheta}=a^{\frac{2}{3}}\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}},\ \frac{b}{\sin\vartheta}=b^{\frac{2}{3}}\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}},$$

and therefore

$$u=AB=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

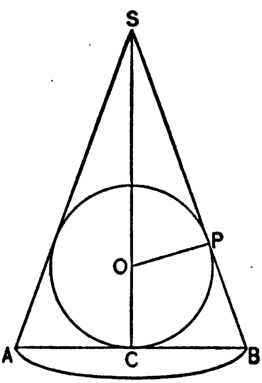
This is the minimum required.

IV. *To find the least cone that can be circumscribed about a given sphere.*

Let $SC=h$ (Fig. 4) be the altitude of the cone, $AC=R$ the radius of its base. Then the expression of its volume V will be

$$V=\frac{h}{3}\pi R^2.$$

Fig. 4



Draw the radius $OP=r$ to the point P where the element BS of the cone touches the sphere. The similar triangles SCB and SOP give us

$$CB : BS :: OP : OS,$$

that is,

$$R : \sqrt{R^2+h^2} :: r : h-r;$$

whence, by developing and reducing, we find

$$R^2=\frac{r^2h}{h-2r}.$$

This value of R^2 substituted in the above equation, gives

$$V=\frac{\pi r^2}{3}\cdot\frac{h^2}{h-2r};$$

whence

$$\frac{dV}{dh} = \frac{\pi r^2}{3} \cdot \frac{(h-2r)2h-h^2}{(h-2r)^2} = 0,$$

from which we obtain

$$h = 4r, \text{ and therefore } R^2 = 2r^2.$$

The volume of the minimum cone will therefore be $V = \frac{8\pi r^3}{3}$. As the *total* surface of this cone is $S = 8\pi r^2$, we see that as its volume is twice that of the sphere, so also is its total surface twice that of the sphere.

V. *A man being in a boat* a *miles distant from the nearest point of the beach, wishes to reach in the shortest time a place* b *miles from that point along the shore. Supposing that he can walk* m *miles an hour, but pull only at the rate of* n *miles an hour, required the place where he must land.*

Let $AB = a$ (Fig. 5), $BC = b$, $BD = x$, D being the point where the man must land. The time employed in rowing from A to D will be $\frac{1}{n}\sqrt{a^2+x^2}$, and the time employed in walking from D to B will be $\frac{1}{m}(b-x)$; so that the total time T employed in the journey will be

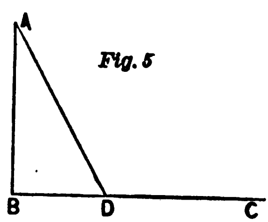


Fig. 5

$$T = \frac{\sqrt{a^2+x^2}}{n} + \frac{b-x}{m}.$$

Differentiating, we find for the minimum

$$\frac{dT}{dx} = \frac{x}{n\sqrt{a^2+x^2}} - \frac{1}{m} = 0, \text{ and } x = \frac{an}{\sqrt{m^2-n^2}}.$$

This value of x gives the distance BD, and the point D where the man must land. The shortest time will be

$$T = \frac{nb + a\sqrt{m^2 - n^2}}{mn}.$$

Make $a = 4$, $b = 8$, $m = 5$, $n = 3$; then $x = 3$, and $T = 2^h\, 40^m$.

VI. *A triangle has a base* b *and a perimeter* 2p. *What must its second and third side be, that the triangle be a maximum?*

Let the second side be denoted by x; the third will then be $2p - b - x$. Its area will then be expressed by the equation

$$A = \sqrt{p(p-b)(p-x)(b+x-p)}.$$

It will be found by the ordinary process that A is a maximum when $x = \frac{2p - b}{2}$; the triangle being isosceles.

VII. *To find the dimensions of the maximum solid cylinder whose total surface is* S.

Let x be the radius of the base, and y the altitude of the cylinder. Its total surface will then be expressed by

$$S = 2\pi x^2 + 2\pi x.y,$$

and its volume by $V = \pi x^2.y$, or, eliminating y, by

$$V = \pi x^2 \frac{S - 2\pi x^2}{2\pi x} = \tfrac{1}{2}(Sx - 2\pi x^3);$$

whence

$$\frac{dV}{dx} = \tfrac{1}{2}(S - 6\pi x^2) = 0,\ x = \sqrt{\frac{S}{6\pi}},\ y = 2\sqrt{\frac{S}{6\pi}},$$

or $y = 2x$. Hence V is a maximum when its altitude is equal to the diameter of its base.

VIII. *To find the greatest isosceles triangle that can be inscribed in a given circle.*

It will be found that it is an equilateral triangle.

IX. *To find the smallest isosceles triangle that can be circumscribed about a given circle.*

It will be found that it is an equilateral triangle.

X. *To find the greatest rectangle that can be inscribed in the ellipse whose semi-axes are* a *and* b.

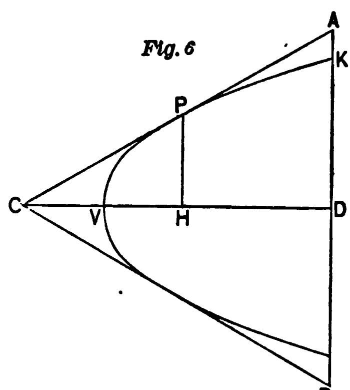
Its base will be $a\sqrt{2}$, and its altitude $b\sqrt{2}$.

XI. *To find the greatest cone that can be cut from a sphere.*

If r is the radius of the sphere, the altitude of the cone will be $\frac{4r}{3}$, and the radius of its base $\frac{2r}{3}\sqrt{2}$.

XII. *To find the greatest segment of a parabola that can be cut from a right cone.*

If h is the altitude of the cone, and r the radius of its base, then the axis of the segment will be $\frac{3}{4}\sqrt{h^2+r^2}$.



XIII. *To find the maximum parabola that can be inscribed in an isosceles triangle having the altitude* h *and the base* b.

Let $VH = x$, and $PH = y$ (Fig. 6) be the coordinates of the point P, where the parabola touches the side CA of the given triangle. The area of the parabola will be $A = \frac{4}{3} VD \times DK$.

As $CV = VH = x$, we have $VD = h - x$. On the other hand,

$\overline{DK}^2 : \overline{PH}^2 :: VD : VH$, or $\overline{DK}^2 : y^2 :: h - x : x$, and therefore

$$DK = y\sqrt{\frac{h-x}{x}}.$$

To eliminate y, we have from the similar triangles ADC and PHC

$PH : AD :: CH : CD$,

$$\text{or } y : \frac{b}{2} :: 2x : h, \text{ or } y = \frac{bx}{h};$$

hence

$$DK = \frac{bx}{h}\sqrt{\frac{h-x}{x}} = \frac{b}{h}\sqrt{x(h-x)}.$$

Accordingly, the equation of the problem becomes

$$A = \frac{4b}{3h}\sqrt{x(h-x)^3}.$$

Hence

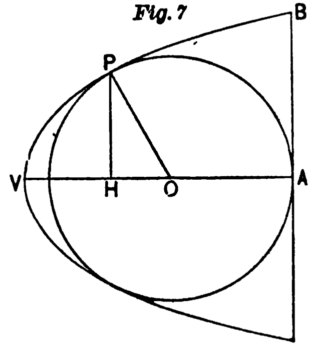
$$\frac{dA}{dx} = \frac{2b}{3h} \cdot \frac{(h-x)^3 - 3x(h-x)^2}{\sqrt{x(h-x)^3}} = 0,$$

and $x = \frac{h}{4}$. Consequently $VD = \frac{3h}{4}$, and $DK = \frac{b}{4}\sqrt{3}$.

XIV. *To find the minimum parabola that can be circumscribed about a circle.*

Let r be the radius of the circle (Fig. 7), $VH = x$, $PH = y$ the co-ordinates of the point P of contact, $VA = a$, and $AB = b$ the terminal co-ordinates, and $2p$ the parameter of the parabola. The area will be $A = \frac{4}{3}ab$.

Fig. 7



Now, $a = VH + HO + OA = x + p + r$, and $b^2 = 2pa$. On the other hand, the triangle POH gives $y^2 = r^2 - p^2 = 2px$, whence $x = \frac{r^2 - p^2}{2p}$; and therefore

$$a = \frac{r^2 - p^2}{2p} + r + p = \frac{r^2 + 2pr + p^2}{2p} = \frac{(p+r)^2}{2p};$$

also $$b^2 = (p+r)^2, \text{ or } b = p + r.$$

Substituting these values of a and b in the expression of A, we have

$$A = \frac{2}{3} \cdot \frac{(p+r)^3}{p};$$

whence

$$\frac{dA}{dp} = \frac{2}{3} \cdot \frac{3p(p+r)^2 - (p+r)^3}{p^2} = 0,$$

and consequently $2p = r$. Hence $a = \frac{9r}{4}$, and $b = \frac{3r}{2}$.

XV. *A cone has a total surface* S. *What must its dimensions be that its volume be a maximum?*

Let h be the altitude of the cone, and r the radius of its base. Its total surface will be $S = \pi r^2 + \pi r \sqrt{h^2 + r^2}$, whence

$$h = \frac{1}{\pi r} \sqrt{S^2 - 2 S\pi r^2}.$$

On the other hand, the volume will be

$$V = \frac{1}{3}\pi r^2 \times h = \frac{r}{3}\sqrt{S^2 - 2S\pi r^2}.$$

Hence

$$\frac{dV}{dr} = \frac{1}{3} \cdot \frac{S^2 - 2S\pi r^2 - 2S\pi r^2}{\sqrt{S^2 - 2S\pi r^2}} = 0,$$

which gives $r = \frac{1}{2}\sqrt{\frac{S}{\pi}}$, and hence $h = \sqrt{\frac{2S}{\pi}}$.

XVI. *To find the maximum cylinder that can be cut from an oblate ellipsoid of revolution, whose semi-axes are* a *and* b.

The radius of the base $= a\sqrt{\frac{2}{3}}$, and the altitude $= b\frac{2}{\sqrt{3}}$

XVII. *What value of* x *will make* $u = \frac{1 - x + x^2}{1 + x - x^2}$ *a minimum?*

The function is a minimum when $x = \frac{1}{2}$.

XVIII. *Through the focus of an ellipse two chords are drawn at right angles. Find when their sum will be a maximum, and when a minimum.*

The solution will be reached through the polar equation of the curve.

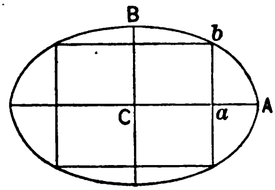
XIX. *Show that* $u = \sin x\,(1 + \cos x)$ *is a maximum when* $x = \frac{\pi}{3}$.

XX. *To find the least ellipse that can be described about a given rectangle.*

Let $2a$ and $2b$ (Fig. 8) be the sides of the given rectangle, and let

$$A^2y^2 + B^2x^2 = A^2B^2$$

Fig. 8



be the equation of the ellipse. In all ellipses we have

$$y^2 : y'^2 :: A^2 - x^2 : A^2 - x'^2 ;$$

hence, if we take $y = B$ and $y' = b$, we shall have $x = 0$, $x' = a$, and

$$B^2 : b^2 :: A^2 : A^2 - a^2, \text{ or } B = \frac{Ab}{\sqrt{A^2 - a^2}} .$$

The area of the ellipse is $E = \pi AB$. Substituting for B its value,

$$E = \pi b \frac{A^2}{\sqrt{A^2 - a^2}} ;$$

hence

$$\frac{dE}{dA} = \pi b \frac{2A(A^2 - a^2) - A^3}{\sqrt{(A^2 - a^2)^3}} = 0,$$

which gives $A = a\sqrt{2}$. And this value substituted in the expression of B gives $B = b\sqrt{2}$. Accordingly the area of the ellipse will be

$$E = \pi a \sqrt{2} \times b \sqrt{2} = 2\pi ab.$$

SCHOLIUM. The theory of maxima and minima, as above developed (No. 28), can be extended to the investigation of the maxima and minima of any function of two independent variables.

Let the function

$$u = f(x, y)$$

be at its maximum or minimum, and let

$$u' = f(x - h, y - k), \quad \text{and } u'' = f(x + h, y + k),$$

be the values of u immediately before and immediately after the maximum or minimum. Then by Taylor's theorem, as extended to two variables (No. 26, *Scholium*), we shall have

$$u' = u - \frac{du}{dx}h - \frac{du}{dy}k + \frac{d^2u}{dx^2}\frac{h^2}{2} + \frac{d^2u}{dxdy}hk + \frac{d^2u}{dy^2}\frac{k^2}{2} - \text{etc.}$$

$$u'' = u + \frac{du}{dx}h + \frac{du}{dy}k + \frac{d^2u}{dx^2}\frac{h^2}{2} + \frac{d^2u}{dxdy}hk + \frac{d^2u}{dy^2}\frac{k^2}{2} + \text{etc.}$$

which may be written as follows:

$$u' - u = - \left(\frac{du}{dx}h + \frac{du}{dy}k\right) + \frac{1}{2}\left(\frac{d^2u}{dx^2}h^2 + 2\frac{d^2u}{dxdy}hk + \frac{d^2u}{dy^2}k^2\right) - \text{etc.}$$

$$u'' - u = + \left(\frac{du}{dx}h + \frac{du}{dy}k\right) + \frac{1}{2}\left(\frac{d^2u}{dx^2}h^2 + 2\frac{d^2u}{dxdy}hk + \frac{d^2u}{dy^2}k^2\right) + \text{etc.}$$

Now, if u be a maximum, then $u > u'$, and $u > u''$, and the two series must be negative. If u be a minimum, then $u < u'$, and $u < u''$, and the two series must be positive. But the two series cannot have equal signs unless the terms

$$-\left(\frac{du}{dx}h + \frac{du}{dy}k\right) \text{ and } + \left(\frac{du}{dx}h + \frac{du}{dy}k\right)$$

disappear; for, when h and k are very small (as they must be in the present case), the sign of the whole series is the same as that of its first term. Hence for the existence of a maximum or of a minimum it is necessary to have

$$\frac{du}{dx}h + \frac{du}{dy}k = 0;$$

and since h and k are independent of each other, we must have separately

$$\frac{du}{dx} = 0 \text{ and } \frac{du}{dy} = 0. \qquad (1)$$

These two equations will give the values of x and y which correspond to a maximum or minimum of the function.

Since the first differential coefficients $\frac{du}{dx}$ and $\frac{du}{dy}$ disappear, the first term of both series will now be the trinomial

$$\frac{1}{2}\left(\frac{d^2u}{dx^2}h^2 + 2\frac{d^2u}{dxdy}hk + \frac{d^2u}{dy^2}k^2\right),$$

which, if negative, will indicate a maximum, and, if positive, a minimum. Making

$$\frac{d^2u}{dx^2} = A, \quad \frac{d^2u}{dxdy} = B, \quad \frac{d^2u}{dy^2} = C,$$

the trinomial will take the form

$$Ah^2 + 2Bhk + Ck^2,$$

or

$$Ak^2\left(\frac{h^2}{k^2} + 2\frac{B}{A}\,\frac{h}{k} + \frac{C}{A}\right),$$

or, by adding and subtracting $\frac{B^2}{A^2}$ and factoring,

$$Ak^2\left\{\left(\frac{h}{k} + \frac{B}{A}\right)^2 + \frac{AC - B^2}{A^2}\right\}. \qquad (2)$$

Now, since h and k are arbitrary, and independent of A and B, we cannot assume $\frac{h}{k} = -\frac{B}{A}$; and therefore the first term $\left(\frac{h}{k} + \frac{B}{A}\right)^2$ of the factor within the brackets cannot be $= 0$, and is always positive. As to the second term, $\frac{AC - B^2}{A^2}$, it is easy to prove that it cannot be negative. For as the sign of (2) must remain unchanged for all the small values of the arbitrary constants h and k, it follows that the value of the expression (2) must not pass through zero. But, if we had $AC - B^2 < 0$, we might choose for h and k such arbitrary values as would give

$$\left(\frac{h}{k} + \frac{B}{A}\right)^2 = \frac{B^2 - AC}{A^2};$$

that is, (2) would pass through zero. Hence the assumption $AC - B^2 < 0$ is inadmissible; and we must therefore have either $AC - B^2 > 0$, or $AC - B^2 = 0$; and thus in both cases the factor within the brackets will be positive. Hence the sign of (2) will be the same as that of the other factor A.

It follows that the existence of a maximum or a minimum cannot be inferred from (2) unless either $AC > B^2$, or $AC = B^2$; that is, unless

$$\frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} > \left(\frac{d^2u}{dxdy}\right)^2, \text{ or } \frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} = \left(\frac{d^2u}{dxdy}\right)^2. \quad (3)$$

The values of x and y which ought to satisfy either the one or the other of these two conditions must, of course, be taken from the equations (1).

The conditions (3) show that the differential coefficients $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ must always be either both negative or both positive; and as the sign of (2), or of the trinomial, is always the same as that of $\frac{d^2u}{dx^2}$, it is plain that when $\frac{d^2u}{dx^2}$ is negative the function u will be a maximum, and when $\frac{d^2u}{dx^2}$ is positive the function u will be a minimum.

But let the student remember that, although the existence of a maximum or minimum cannot be tested by (2) when the trinomial is $=0$, yet even in this case there may be a maximum or minimum: but it must then be determined by the sign of the fourth differential coefficients, after having ascertained that the third differential coefficients, which have opposite signs, reduce to zero, as the theory (No. 28) requires.

EXAMPLE I. A cistern which is to contain a certain quantity of water is to be constructed in the form of a rectangular parallelopipedon. Determine its form, so that the smallest possible expense shall be incurred in lining the internal surface.

SOLUTION. Let $a^3 =$ its content, $x =$ length, $y =$ breadth, and therefore $\frac{a^3}{xy} =$ depth. The total surface u will be

$$u = xy + 2\frac{a^3}{x} + 2\frac{a^3}{y} = \text{a minimum.}$$

Differentiating first with regard to x, then with regard to y, we find

$$\frac{du}{dx} = y - \frac{2a^3}{x^2} = 0, \qquad \frac{du}{dy} = x - \frac{2a^3}{y^2} = 0;$$

hence

$$x^2y = 2a^3 = y^2x, \text{ and } x = y = a\sqrt[3]{2};$$

and therefore the base must be a square. The depth will be

$$\frac{a^3}{xy} = \frac{a^3}{a^2\sqrt[3]{4}} = a\frac{\sqrt[3]{2}}{\sqrt[3]{8}} = \frac{1}{2}a\sqrt[3]{2};$$

and therefore the depth must be equal to half the length or breadth. Since

$$\frac{d^2u}{dx^2}=\frac{4a^3}{x^3}=\frac{4a^3}{2a^3}=2,\ \frac{d^2u}{dy^2}=2,\ \frac{d^2u}{dxdy}=1,$$

the first of conditions (3) is satisfied, and u is a minimum.

EXAMPLE II. Find the values of x and y which shall make the function

$$u=x^4+y^4-4axy^2$$

a maximum or a minimum.

SOLUTION. Here we find

$$\frac{du}{dx}=4x^3-4ay^2=0, \qquad \frac{du}{dy}=4y^3-8axy=0\,;$$

hence

$$x^3=ay^2,\ y^2=2ax,\ x^3=2a^2x,\ x^2=2a^2$$

$$x=\pm\,a\sqrt{2},\ y^2=2a^2\sqrt{2}=a^2\sqrt{8},\ y=\pm\,a\sqrt[4]{8}.$$

And again,

$$\frac{d^2u}{dx^2}=12x^2=24a^2,\ \ \frac{d^2u}{dy^2}=12y^2-8ax=16a^2\sqrt{2},$$

$$\frac{d^2u}{dxdy}=-8ay,\ \ \left(\frac{d^2u}{dxdy}\right)^2=64a^2y^2=64a^4\sqrt{8}\,;$$

and therefore the first condition (3) is satisfied; and as the sign of $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ is positive, $x=\pm\,a\sqrt{2}$ and $y=\pm\,a\sqrt[4]{8}$ make u a minimum.

In this example, the equations $\frac{du}{dx}=0$ and $\frac{du}{dy}=0$ are also satisfied by taking $x=0$ and $y=0$. With these values of x and y we find $\frac{d^2u}{dx^2}=0$ and $\frac{d^2u}{dy^2}=0$. And since in this case the third differential coefficients are

$$\frac{d^3u}{dx^3}=24x=0,\ \ \frac{d^3u}{dy^3}=24y=0,$$

and the fourth differential coefficients

$$\frac{d^4u}{dx^4}=24,\ \ \frac{d^4u}{dy^4}=24,$$

are positive, we conclude that the values $x = 0$ and $y = 0$ correspond to another minimum.

Remark. When u is a function of three independent variables, the conditions of its maxima and minima are determined by a process analogous to the preceding, but which is based on the extension of Taylor's formula to a function of three variables, and is too long to be inserted here. The result, however, of such an investigation is simple enough. If a function

$$u = f(x, y, z)$$

has any maximum or minimum, it must give

$$\frac{du}{dx} = 0, \ \frac{du}{dy} = 0, \ \frac{du}{dz} = 0,$$

and the values of x, y, z found from these equations must satisfy the condition

$$\left\{ \frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} - \left(\frac{d^2u}{dxdy}\right)^2 \right\} \cdot \left\{ \frac{d^2u}{dx^2} \cdot \frac{d^2u}{dz^2} - \left(\frac{d^2u}{dxdz}\right)^2 \right\}$$
$$> \left(\frac{d^2u}{dydz} \cdot \frac{d^2u}{dx^2} - \frac{d^2u}{dxdy} \cdot \frac{d^2u}{dxdz}\right)^2.$$

The function will be a maximum if the two factors within brackets in the first member of this inequality are negative; but a minimum if they are positive.

Values of Functions which assume an Indeterminate Form.

31. It sometimes happens that in giving to the variable a certain value, the function assumes one of the forms

$$\frac{0}{0}, \ \frac{\infty}{\infty}, \ 0 \times \infty, \ x - \infty, \ \frac{1}{0 \times \infty}.$$

Thus the fraction $\frac{x^4 - 1}{x^2 - 1}$, when $x = 1$, takes the form $\frac{0}{0}$, though its real value is 2. The real values of functions that assume the form $\frac{0}{0}$ can be found by the following process.

Let z and y be functions of x, and let $u = \frac{z}{y}$ become $\frac{0}{0}$ when $x = a$. Clearing of fractions, and differentiating, we have

$$udy + ydu = dz;$$

but when $x = a$, the term ydu disappears. Hence $(udy)_a = (dz)_a$, or

$$u_a = \left(\frac{dz}{dy}\right)_a.$$

Accordingly, the value of u, when it takes the form $\frac{0}{0}$ for $x = a$, will be found by differentiating separately the numerator and the denominator of its expression, and substituting in the resulting fraction the value $x = a$.

If $\left(\frac{dz}{dy}\right)_a$ were again of the form $\frac{0}{0}$, we would apply again the same process of differentiation, and we would obtain

$$u_a = \left(\frac{d^2z}{d^2y}\right)_a,$$

and, if necessary, we might continue the same process until a determinate value is reached. Thus the fraction $\frac{x - \sin x}{x^3}$ becomes $\frac{0}{0}$ when $x = 0$; but by the process just explained we successively obtain

$$\left[\frac{x-\sin x}{x^3}\right]_0 = \left[\frac{1-\cos x}{3x^2}\right]_0 = \left[\frac{\sin x}{6x}\right]_0 = \left[\frac{\cos x}{6}\right]_0 = \frac{1}{6}.$$

When the function, for a certain value of x, assumes the form $\frac{\infty}{\infty}$, or the form $0 \times \infty$, or the form $\frac{1}{0 \times \infty}$, its real value may be found by first reducing it to the form $\frac{0}{0}$, and then applying the process above explained. The reduction is easily obtained by remembering that $\infty = \frac{1}{0}$. Sometimes this reduction is not needed. Thus the function $\frac{\log x}{x-a}$, which takes the form $\frac{\infty}{\infty}$ when $x = \infty$, will give immediately

$$\left[\frac{\log x}{x-a}\right]_\infty = \left[\frac{\frac{1}{x}}{1}\right]_\infty = 0.$$

The form $\infty - \infty$ may also be reduced to the form $\frac{0}{0}$. For let v and w be two functions of x, which for a certain value of x become infinite. Then the function $u = v - w$ becomes $\infty - \infty$ for that value of x. But we have

$$u = v - w = v\left(1 - \frac{w}{v}\right) = \frac{1 - \frac{w}{v}}{\frac{1}{v}};$$

and when $v = \infty$ and $w = \infty$, the function will take the form $\frac{0}{0}$, provided we have $1 - \frac{w}{v} = 0$. If this condition were not fulfilled, then the equation $u = v\left(1 - \frac{w}{v}\right)$ would make the function infinite.

Assume

$$u = \frac{\pi}{2} \sec x - x \tan x;$$

when $x = \frac{\pi}{2}$, the function takes the form $\infty - \infty$. But we may write

$$(u)_{\frac{\pi}{2}} = \left(\frac{\pi}{2} \sec x - x \tan x\right)_{\frac{\pi}{2}} = \left(\frac{1 - \frac{x \tan x}{\frac{\pi}{2} \sec x}}{\frac{1}{\frac{\pi}{2} \sec x}}\right)_{\frac{\pi}{2}} = \frac{0}{0};$$

and from this we shall obtain $(u)_{\frac{\pi}{2}} = 1$.

32. The indeterminate forms 0^0, ∞^0, 1^∞, can be reduced by the following method. Let v and w be two functions of x of such a nature that, when $x = a$, they cause the expression $u = v^w$ to assume one of the forms 0^0, ∞^0, 1^∞. Since $v = e^{\log v}$ (e being the base of the Napierian logarithms), we shall have $v^w = e^{w \log v}$. Now, the exponent $w \log v$ in each of the three proposed cases takes the form $0 \times \infty$, which can be reduced to $\frac{0}{0}$, as we have explained.

EXAMPLES.

1. $\left[\frac{e^x - e^{-x}}{\log(1 + x)}\right]_0 = 2,$

2. $\left[\frac{x^n}{e^x}\right]_\infty = 0,$

3. $[x \log x]_0 = 0$,

4. $\left[\dfrac{x^2 \tan x}{1 + \tan x}\right]_{\frac{\pi}{2}} = \dfrac{\pi^2}{4}$,

5. $\left[(1 - x) \tan \dfrac{\pi x}{2}\right]_1 = \dfrac{2}{\pi}$,

6. $\left[\tan \dfrac{\pi x}{2a} \log \left(2 - \dfrac{x}{a}\right)\right]_a = \dfrac{2}{\pi}$,

7. $\left(\dfrac{x - \sqrt{2x^2 - a^2}}{2x - \sqrt{5x^2 - a^2}}\right)_a = 2$,

8. $[x^x]_0 = 1$,

9. $\left[\left(\dfrac{1}{x}\right)^{\sin x}\right]_0 = 1$,

10. $\left[\dfrac{1}{x \cot x}\right]_0 = 1$

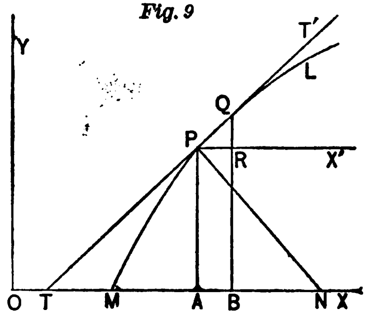
11. $\left[\dfrac{a^x - b^x}{x}\right]_0 = \log a - \log b$.

SECTION III.

INVESTIGATIONS ABOUT PLANE CURVES.

33. Let $y = f(x)$ be the equation of a plane curve MPL (Fig. 9), and let $x = OA$ and $y = PA$ be the co-ordinates of a point P of the curve. Draw PT' tangent to the curve at P, and PX' parallel to the axis OX; and make the angle $T'PX' = \vartheta$.

Fig. 9



Let the point Q be consecutive to the point P. The infinitesimal increment PQ of the curve entails an increment AB of the abscissa x, and an increment QR of the ordinate y. If then we represent by s the portion MP of the curve, then $PQ = ds$, whilst $AB = dx$, and $QR = dy$. Now, we have

$$AB = PQ \cos \vartheta,\ QR = PQ \sin \vartheta,\ \overline{PQ}^2 = \overline{PR}^2 + \overline{QR}^2,$$

hence

$$dx = ds \cos \vartheta,\ dy = ds \sin \vartheta,\ ds = \sqrt{dx^2 + dy^2},$$

and

$$\frac{dx}{ds} = \cos \vartheta,\ \frac{dy}{ds} = \sin \vartheta,\ \frac{dy}{dx} = \tan \vartheta.$$

As the tangent PT is but a secant which meets the curve at two *consecutive* points, it follows that the tangent and the curve have a common infinitesimal element PQ, and that the angle which the element PQ of the curve makes with the axis OX is identical with the angle ϑ made by the tangent at P with the same axis.

The trigonometric tangent of the angle ϑ is taken as a measure of *the slope* of the curve at the point P, and, as $\tan\vartheta = \frac{dy}{dx}$, *the differential coefficient of the ordinate of any point of the curve is the measure of the slope of the curve at that point.*

Tangents, Normals, etc.

34. The equation of a straight line passing through two given points of a curve, whose co-ordinates are x', y', and x'', y'', is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x').$$

When the two points are consecutive, as P and Q (Fig. 9), then $y'' - y' = dy'$, and $x'' - x' = dx'$; and the equation becomes

$$y - y' = \frac{dy'}{dx'}(x - x').$$

This is the equation of the tangent to the curve at the point P, whose co-ordinates are denoted by y', x'.

Making $y = 0$, we find for the point T, where the tangent meets the axis OX,

$$x = x' - y'\frac{dx'}{dy'}.$$

The subtangent AT is evidently $= x' - x = y'\frac{dx'}{dy'}$.

35. The normal being perpendicular to the tangent at the point of contact, its equation can be derived from that of the tangent by substituting $-\frac{dx'}{dy'}$ instead of $+\frac{dy'}{dx'}$. Hence

$$y - y' = -\frac{dx'}{dy'}(x - x')$$

is the equation of the normal to the curve at the point (x', y').

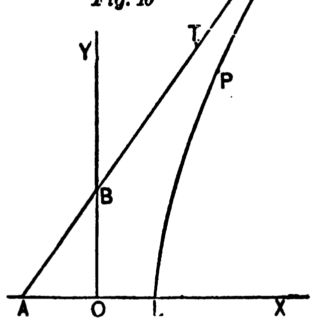
Making $y = 0$, we find for the point N, where the normal meets the axis OX,

$$x = x' + y'\frac{dy'}{dx'}.$$

The subnormal AN is evidently equal to $x - x'$ $= y'\frac{dy'}{dx'}$.

36. An *asymptote* to a curve is a line that continually approaches the curve and becomes tangent to it at an infinite distance. Such a line will, of course, cut either one or both the co-ordinate axes at a *finite* distance from the origin.

Fig. 10



Let the straight line AT (Fig. 10) be an asymptote to the curve LP. Since AT, when infinitely prolonged, is a tangent to the curve, its equation will be of the form

$$y - y' = \frac{dy'}{dx'}(x - x'),$$

x' and y' being the co-ordinates of the point of contact infinitely distant from the origin. This equation, when $x = 0$, gives

$$y = y' - \frac{dy'}{dx'}x' = OB;$$

and, when $y = 0$, it gives

$$x = x' - \frac{dx'}{dy'}y' = OA.$$

If the values of OA and OB obtained from these equations are both finite, the asymptote will intersect both co-ordinate axes; if one of the two values is infinite, the asymptote will be parallel to one of the axes; if both values are zero, the asymptote will pass through the origin, and its direction will be determined by the value of $\frac{dy'}{dx'}$. If both OA and OB are infinite, the curve has no asymptote, as no place can be found for it in the plane of the co-ordinate axes.

37. Let us inquire, for example, whether the curve

$$y^2 = 2x + 3x^2$$

has any asymptote. We find by differentiation

$$\frac{dy}{dx} = \frac{1 + 3x}{y} = \frac{1 + 3x}{\pm\sqrt{2x + 3x^2}}.$$

This value being put in the expressions for OA and OB will give, after reduction,

$$OA = -\frac{x}{1+3x},\quad OB = \frac{x}{\pm\sqrt{2x+3x^2}}.$$

And now make $x = \infty$. Then

$$OA = -\frac{1}{3},\quad OB = \pm\frac{1}{\sqrt{3}}.$$

Hence the curve has two asymptotes, one of which intersects the axes at the distance $x = -\frac{1}{3}$, $y = \frac{1}{\sqrt{3}}$, and the other at the distance $x = -\frac{1}{3}$, $y = -\frac{1}{\sqrt{3}}$.

Again, let us inquire whether the curve

$$y = \log x$$

has any asymptote. Here we have

$$\frac{dy}{dx} = \frac{1}{x};$$

hence

$$OA = x\ (1 - \log x),\quad OB = \log x - 1.$$

The assumption $y = \infty$ gives both OB and OA infinite, and cannot correspond to an asymptote; but the assumption $y = -\infty$ gives $x = 0$, whence we obtain (No. 31)

$$OA = [x(1 - \log x)]_0 = \left[\frac{1 - \log x}{\frac{1}{x}}\right]_0$$

$$= \left[\frac{-\frac{1}{x}}{-\frac{1}{x^2}}\right]_0 = [x]_0 = 0;$$

and therefore the curve has an asymptote which passes through the origin, and makes with the axis OX an angle whose tangent is $\frac{dy}{dx} = \frac{1}{x} = \infty$; that is, the asymptote is at right angles to OX, and coincides with the axis OY.

Let us inquire also whether the curve

$$y = \tan x$$

has any asymptote. Here we have

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} = 1 + \tan^2 x = 1 + y^2;$$

hence

$$OA = \tan^{-1} y - \frac{y}{1+y^2}, \quad OB = \tan x - \frac{x}{\cos^2 x}.$$

Make $y = \infty$. Then

$$OA = \frac{\pi}{2}, \quad OB = \infty.$$

Hence the curve has an asymptote parallel to the axis OY, at a distance $\frac{\pi}{2}$ from the origin. As the value $y = \infty$ corresponds not only to $x = \frac{\pi}{2}$, but also to $x = \frac{3\pi}{2}, = \frac{5\pi}{2}, = \frac{(2n+1)\pi}{2}$, it follows that there will be an endless series of asymptotes, all parallel to the axis OY, and all at a distance π from each other.

To find whether the parabola $y^2 = 2px$ has asymptotes, let us put its differential coefficient $\frac{dy}{dx} = \frac{p}{y}$ in the expressions for OA and OB. We shall find, after reduction,

$$OA = -x, \quad OB = \frac{y}{2}.$$

Making $y = \infty$, we shall have also $x = \infty$; and thus both OA and OB will be infinite. Hence this curve has no asymptotes.

In the hyperbola represented by the equation

$$a^2y^2 - b^2x^2 = -a^2b^2$$

we have

$$\frac{dy}{dx} = \frac{b^2x}{a^2y};$$

hence, by substitution and reduction,

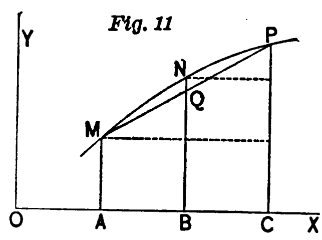
$$OA = \frac{a^2}{x}, \quad OB = -\frac{b^2}{y}.$$

Making $y = \pm \infty$, we shall have also $x = \pm \infty$; and therefore $OA = 0$ and $OB = 0$. Hence the hyperbola has two asymptotes passing through the origin of co-ordinates, and making with the axis OX two supplementary angles, of which the acute one corresponds to x and y affected by equal signs, and the obtuse one to x and y affected by opposite signs.

Direction of Curvature.

38. Let M, N, P (Fig. 11) represent three consecutive points of a plane curve $y = f(x)$. If we draw the chord MP, and if the point N lies above the chord, the curve is said to turn its *concavity* downwards; if the point N were to fall beneath the chord, the curve would be said to turn its *convexity* downwards.

Fig. 11



Let y, y', y'' be the equidistant ordinates of the points M, N, P. Since y' is consecutive to y, and y'' consecutive to y', we have

$$y' = y + dy, \quad y'' = y' + dy' = y + dy + dy'.$$

If the curve is *concave* downwards, as it is in our diagram, then $BN > BQ$. But $BN = y'$, and $BQ = \frac{1}{2}(y + y'')$; therefore

$$y' > \tfrac{1}{2}(y + y''), \text{ or } 2y' > y + y'',$$

that is,

$$2(y + dy) > y + y + dy + dy'$$

and, by reduction, $dy > dy'$, or $dy' - dy < 0$, or $d^2y < 0$, and consequently $\frac{d^2y}{dx^2} < 0$.

If the curve is *convex* downwards, then $BN < BQ$, and consequently $dy < dy'$; hence $dy' - dy > 0$, or $d^2y > 0$, and $\frac{d^2y}{dx^2} > 0$.

Accordingly, *whenever the second differential coefficient of the function is negative, the curve is concave downwards; and whenever the same coefficient is positive, the curve is convex downwards.*

If, in the equation of the curve, we take y as independent, we can show, by a reasoning similar to the above, that *the curve turns its convexity or its concavity to the left, according as the second differential coefficient* $\frac{d^2x}{dy^2}$ *is positive or negative.*

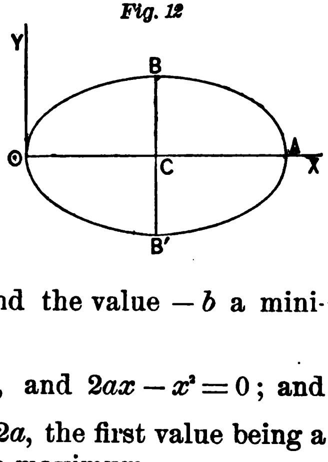
39. If, for a certain value of x, the first differential coefficient $\frac{dy}{dx}$ becomes $= 0$, and if the second differential coefficient $\frac{d^2y}{dx^2}$ be negative, that value of x will answer to a maximum; whereas if $\frac{d^2y}{dx^2}$ be positive, that value of x will answer to a minimum, as we have seen (No. 28). It is plain, then, that any maxima and minima can be *graphically* exhibited as ordinates to the culminating points of a curve.

When $\frac{dy}{dx}=0$, the tangent to the curve at the point of the maximum or of the minimum is parallel to the axis of x. As, however, there may also be a maximum or a minimum for $\frac{dy}{dx}=\infty$, that is, for $\frac{dx}{dy}=0$, the tangent may also be parallel to the axis of y.

Thus in the ellipse (Fig. 12) whose equation is

$$y^2=\frac{b^2}{a^2}(2ax-x^2),$$

Fig. 12



we have

$$\frac{dy}{dx}=\pm\frac{b(a-x)}{a\sqrt{2ax-x^2}}.$$

If $\frac{dy}{dx}=0$, then $x=a$, and $y=\pm b$, the value $+b$ being a maximum, and the value $-b$ a minimum.

If $\frac{dy}{dx}=\infty$, then $\frac{dx}{dy}=0$, and $2ax-x^2=0$; and accordingly $x=0$, or $x=2a$, the first value being a minimum and the second a maximum.

The double sign which affects the expression for $\frac{dy}{dx}$, equally affects the expression for $\frac{d^2y}{dx^2}$; and therefore while one half of the curve turns its concavity downwards, the other half turns it upwards. The value $\frac{dy}{dx} = 0$ expresses the fact that the tangents at the points B and B' are parallel to the axis OX, and the value $\frac{dy}{dx} = \infty$ expresses the fact that the tangents at O and at A are perpendicular to OX, that is, parallel to OY.

Singular Points.

40. A singular point is a point at which the curve presents some peculiarity not common to other points. Four classes of singular points are particularly remarkable; *points of inflection*, *cusps*, *multiple points*, and *conjugate points*.

A *point of inflection* is a point at which the curvature changes from concave downwards to convex downwards, or *vice-versa*. This change requires that the second differential coefficient of the function should change from $-$ to $+$, or from $+$ to $-$; hence, at a point of inflection, the second differential coefficient must be either $= 0$ or $= \infty$. If, then, the co-ordinates of the curve are such that, for values immediately preceding and following, the second differential coefficient has contrary signs, they are the co-ordinates of a point of inflection.

Thus, the curve

$$y = b + (x - a)^3$$

gives

$$\frac{dy}{dx} = 3(x - a)^2, \quad \frac{d^2y}{dx^2} = 6(x - a).$$

When $x = a$, then $\frac{d^2y}{dx^2} = 0$. When $x < a$, then

$\frac{d^2y}{dx^2} < 0$, and when $x > a$, then $\frac{d^2y}{dx^2} > 0$. Hence at $x = a$ there is a point of inflection, whose co-ordinates are $x = a$, $y = b$; and since, at this point, we have $\frac{dy}{dx} = 0$, the tangent passing through it is parallel to the axis of abscissas, as in Fig. 13.

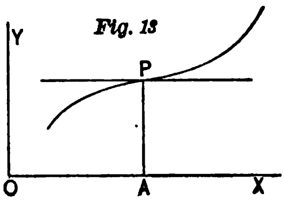


Fig. 13

The curve

$$y = a\left(1 - \cos\frac{x}{c}\right)$$

gives

$$\frac{dy}{dx} = \frac{a}{c}\sin\frac{x}{c}, \quad \frac{d^2y}{dx^2} = \frac{a}{c^2}\cos\frac{x}{c}.$$

When $x = \frac{1}{2}\pi c$, then $\frac{d^2y}{dx^2} = 0$. When $x < \frac{1}{2}\pi c$, then $\frac{d^2y}{dx^2} > 0$, and when $x > \frac{1}{2}\pi c$, then $\frac{d^2y}{dx^2} < 0$. Hence at $x = \frac{1}{2}\pi c$ there is a point of inflection, whose co ordinates are $x = \frac{1}{2}\pi c$ and $y = a$.

The slope of the tangent through it is $\frac{dy}{dx} = \frac{a}{c}$, and is a maximum. The curve has a minimum ordinate ($y = 0$) at the origin, and a maximum ordinate ($y = 2a$) at the point where $x = \pi c$.

The curve (Fig. 14) extends indefinitely to the right and to the left of the origin; for, making $x = \pm n\pi c$, the values

$$n = \tfrac{1}{2}, \ = \tfrac{3}{2}, \ = \tfrac{5}{2}, \ \ldots$$

will give a series of points of inflection, $P, Q, R, \ldots$ whilst the values

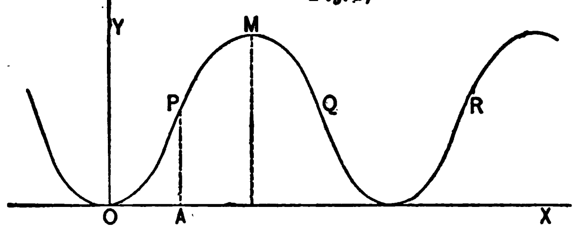
$$n = 1, \ = 3, \ = 5, \ . \ . \ .$$

will give a series of minimum ordinates ($y = 0$), and the values

$$n = 2, \ = 4, \ = 6, \ . \ . \ .$$

will give a series of equal maximum ordinates ($y = 2a$).

Fig. 14



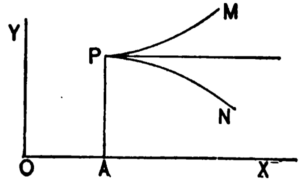
The slopes of the tangents at the points of inflection will be alternately

$$\frac{dy}{dx} = +\frac{a}{c} \text{ and } \frac{dy}{dx} = -\frac{a}{c}.$$

41. A *cusp* is a point at which two branches of a curve terminate, being tangential to each other. There are two species of cusps. In the first species the two branches lie on opposite sides of a tangent drawn at their common point; in the second species both branches lie upon the same side of the tangent.

Thus, the curve of the equation $y = b \pm (x - a)^{\frac{3}{2}}$ will have a cusp of the first species (Fig 15); for we find

Fig. 15



$$\frac{dy}{dx} = \pm \tfrac{3}{2}(x - a)^{\frac{1}{2}},$$

and

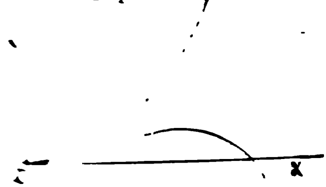
$$\frac{d^2y}{dx^2} = \pm \tfrac{3}{4}(x - a)^{-\frac{1}{2}};$$

and when $x = a$, we have $\frac{dy}{dx} = 0$, and $\frac{d^2y}{dx^2} = \pm x$.

The curve has, therefore, two branches, PM and PN, the one convex, the other concave, downwards, which meet tangent ... at the point P, whose co-ordinates are ... $x = a$, and which do not extend to the left of P, because, when $x < a$, y is imaginary.

The curve of the equation $y = x^2 \pm x^{\frac{5}{2}}$ will have a cusp of the second species [illegible]

Fig. [illegible]



[illegible]

and when [illegible] both branches [illegible] positive, both [illegible] upwards. The [illegible] cannot be [illegible]

[illegible] the lower branch [illegible] corresponds [illegible] the same lower branch. [illegible] value cor- [illegible] This lower branch [illegible] from the origin. [illegible] that there is a cusp whenever the differential coefficient has two values which become identical at the point where the curve stops.

42. A *multiple point* is a point where two or more branches of a curve intersect, or touch each other. If the branches intersect, $\frac{dy}{dx}$ will have as many values at that point as there are branches; if they are tangent, these values will be equal.

Thus in the curve

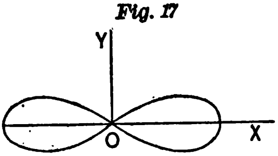
$$y = \pm x\sqrt{1 - x^2},$$

we have

$$\frac{dy}{dx} = \pm \sqrt{1 - x^2} \mp \frac{x^2}{\sqrt{1 - x^2}};$$

and when $x = 0$, then $\frac{dy}{dx} = \pm 1$, which shows that the curve has two branches *intersecting* at the origin (Fig. 17). As x cannot exceed ± 1, the curve is limited in both directions. The maximum and minimum ordinates correspond to $\frac{dy}{dx} = 0$, in which case we have

Fig. 17



$$\sqrt{1 - x^2} = \frac{x^2}{\sqrt{1 - x^2}},$$

whence

$$x = \pm \frac{1}{\sqrt{2}};$$

and the corresponding ordinate will then be

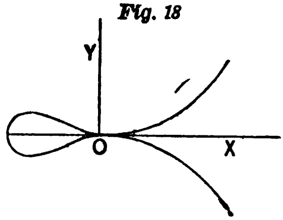
$$y = \pm \tfrac{1}{2}.$$

In the curve $y = \pm x^2 \sqrt{x+1}$ we have

$$\frac{dy}{dx} = \pm \frac{5x^2 + 4x}{2\sqrt{x+1}}.$$

When $x = 0$, then $\frac{dy}{dx} = 0$. Hence the origin O (Fig. 18) is a multiple point, where the two branches are *tangent* to each other and to the axis OX. The curve cuts the axis at the point $x = -1$, where $\frac{dy}{dx} = \infty$.

Fig. 18



In the curve

$$x^4 + 2x^2y - y^3 = 0$$

we have

$$x = \pm \sqrt{-y \pm y\sqrt{y+1}},$$

whence

$$\frac{dy}{dx} = \frac{\pm 4\sqrt{y+1}\sqrt{-y \pm y\sqrt{y+1}}}{\pm 3y \pm 2 - 2\sqrt{y+1}}.$$

This expression, when $y = 0$, becomes either $= \frac{0}{0}$ or $= 0$, according as we take the upper or the lower signs. But the real values of $\frac{dy}{dx}$, when $y = 0$, can be found by putting the equation of the curve under the form

$$x + 2\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^3 = 0,$$

which, when $x = 0$ and $y = 0$, may be written thus

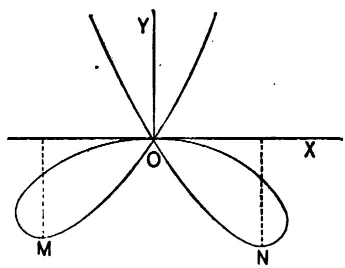
$$0 + 2\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3 = 0,$$

it being obvious that $\frac{dy}{dx}$ is nothing else than the ratio $\frac{y}{x}$, when these quantities are becoming each $=0$. We thus find

$$\frac{dy}{dx}=0, \text{ and } \frac{dy}{dx}=\pm\sqrt{2}.$$

These values show that one branch of the curve is tangent to the axis OX at the origin (Fig. 19), while a second branch, at the origin, makes with the same axis an angle whose tangent is $=\sqrt{2}$, and a third branch, at the origin, makes with the same axis an angle whose tangent is $=-\sqrt{2}$. Hence the curve has a triple point at the origin, with two symmetrical loops under the axis of x.

Fig. 19



Its lowest ordinate is $y=-1$, corresponding to $x=\pm 1$, for which we have $\frac{dy}{dx}=0$. Hence the curve, at the corresponding points M and N, is parallel to the axis OX.

43. A *conjugate point* is a point whose co-ordinates satisfy the equation of the curve, but which is *isolated*, and therefore has no consecutive points. Thus the equation

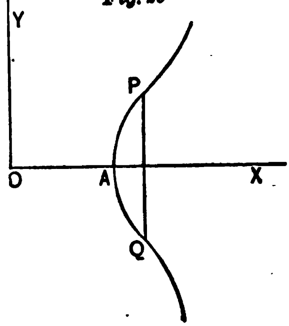
$$y^2=x^2(x-a)$$

is satisfied by the co-ordinates $x=0$, $y=0$, but no other value of x less than a gives a point of the curve. Hence the point $x=0$, $y=0$ is an isolated point. The differential coefficient of the equation, which is

$$\frac{dy}{dx} = \pm \frac{3x - 2a}{2\sqrt{x - a}},$$

shows that the curve (Fig. 20) has two branches cutting the axis OX perpendicularly at the point $x = a$. It shows also that when $x=0$, $\frac{dy}{dx}$ is imaginary; which expresses the obvious fact that a solitary point can form no angle whatever with the axis of x.

Fig. 20



The second differential coefficient of the equation, which is

$$\frac{d^2y}{dx^2} = \pm \frac{6(x - a) - 3x + 2a}{4\sqrt{(x - a)^3}},$$

shows that the curve has two symmetrical points of inflection, P and Q, corresponding to the abscissa

$$x = \frac{4a}{3}.$$

If a curve has a series of conjugate points, this series is called a *branche pointillée*, or a *dotted branch*. For example, in the curve

$$y^2 = x \sin^2 x,$$

every positive value of x will give two values for y; but when x is negative, y becomes imaginary, except when x is a multiple of π, in which case $y = 0$. There will then be an indefinite series of conjugate points lying on the axis of x at equal distances, and forming a *dotted branch.*

As a second example, take the curve

$$y = ax^2 \pm \sqrt{x(1-\cos x)}.$$

Here for every positive value of x we have two values of y, and consequently two points, except when $\cos x = 1$, in which case the two points reduce to one. Hence the curve, owing to the periodic coincidence of such points, will consist of a series of loops not unlike the links of a chain, having for a diametral curve the parabola $y = ax^2$.

But for all negative values of x the ordinate is imaginary, except when $\cos x = 1$; and in these cases we have a series of isolated points situated on the left branch of the diametral curve.

Order of Contact, Osculation.

44. Let $y = \varphi(x)$ and $y_1 = f(x)$ represent two curves. If x be an abscissa common to both curves, and if this abscissa receive an infinitesimal increment h, we shall have (No. 25)

$$\varphi(x+h) = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{2} + \frac{d^3y}{dx^3}\frac{h^3}{2.3} + \ldots$$

$$f(x+h) = y_1 + \frac{dy_1}{dx}h + \frac{d^2y_1}{dx^2}\frac{h^2}{2} + \frac{d^3y_1}{dx^3}\frac{h^3}{2.3} + \ldots$$

Now, if in these equations we have $y = y_1$, the curves will have a common point: if we have also $\frac{dy}{dx} = \frac{dy_1}{dx}$, they will have two common points, *i. e.*, will be tangent to each other, their contact being of the *first order*. If we have also $\frac{d^2y}{dx^2} = \frac{d^2y_1}{dx^2}$, the curves will have three common points, and their contact will be more intimate than in the preceding case: it will be a contact of the *second order*. If we had also $\frac{d^3y}{dx^3} = \frac{d^3y_1}{dx^3}$, $\frac{d^4y}{dx^4} = \frac{d^4y_1}{dx^4}$, . . . the curves would have a contact of the third, fourth . . . order.

The order of the contact corresponds to the number of successive differential coefficients which are common to both lines.

Any line which at a given point of a curve has a higher order of contact than any other line of its own species, is called an *Osculatrix*. Thus, an *osculatory circle* is a circle that has a higher order of contact than any other circle.

45. If a curve $y = f(x)$ has an osculatrix of the n^{th} order, then the osculatrix, at the point (x', y') of osculation, must have the same co-ordinates x', y', and n successive differential coefficients identical with those of the curve; in other terms, it must satisfy $n+1$ conditions. Hence if a curve of a certain species is to be made an osculatrix of the n^{th} order, its general equation must contain $n+1$ arbitrary constants, by the determination of which the $n+1$ conditions may be satisfied.

For example, since the general equation of a straight line is

$$y = ax + b,$$

and contains *two* constants, the straight line can be made an osculatrix of the *first* order. For when the function $y = ax + b$, and its derivative $\frac{dy}{dx} = a$, are assumed respectively equal to the functions y' and $\frac{dy'}{dx'}$ of the given curve at the point (x', y'), we have

$$a = \frac{dy'}{dx'}, \text{ and } b = y' - \frac{dy'}{dx'} x';$$

and thus the equation of our straight line will become

$$y = \frac{dy'}{dx'} x + y' - \frac{dy'}{dx'} x',$$

that is,

$$y - y' = \frac{dy'}{dx'} (x - x'),$$

which is the well-known equation of a tangent at a point (x', y') of the curve. This tangent is an *osculatrix.*

Thus also, since the general equation of the circle

$$(x - \alpha)^2 + (y - \beta)^2 = R^2$$

contains *three* arbitrary constants, the circle can be made an osculatrix of the *second* order. For, when the function y drawn from this equation, and its derivatives

$$\frac{dy}{dx} = -\frac{x - \alpha}{y - \beta}, \quad \frac{d^2y}{dx^2} = -\frac{1 + \frac{dy^2}{dx^2}}{y - \beta},$$

are assumed respectively equal to the functions y', $\frac{dy'}{dx'}$, $\frac{d^2y'}{dx'^2}$ of the given curve at the point (x', y'), we have the equations

$$R^2 = (x' - \alpha)^2 + (y' - \beta)^2,$$

$$\frac{x' - \alpha}{y' - \beta} = -\frac{dy'}{dx'}, \text{ and } \frac{1 + \frac{dy'^2}{dx'^2}}{y' - \beta} = -\frac{d^2y'}{dx'^2};$$

from which we obtain

$$x' - \alpha = -\frac{dy'}{dx'}(y' - \beta), \quad y' - \beta = -\frac{1 + \frac{dy'^2}{dx'^2}}{\frac{d^2y'}{dx'^2}},$$

and finally

$$R = \pm \frac{\left(1 + \frac{dy'^2}{dx'^2}\right)^{\frac{3}{2}}}{\frac{d^2y'}{dx'^2}}.$$

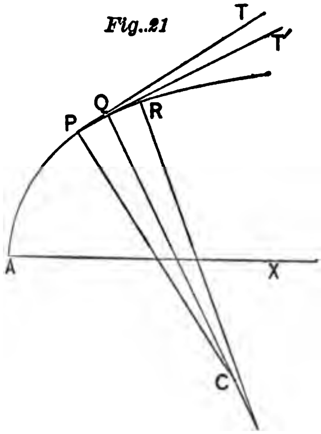
Such is the expression of the radius of the osculatory circle. The co-ordinates of its centre are

$$\alpha = x' - \frac{1 + \frac{dy'^2}{dx'^2}}{\frac{d^2y'}{dx'^2}} \cdot \frac{dy'}{dx'}, \quad \beta = y' + \frac{1 + \frac{dy'^2}{dx'^2}}{\frac{d^2y'}{dx'^2}}.$$

Measure of Curvature.

46. Curvature is *the ratio between the change of direction and the length over which such a change takes place.* Thus, if ϑ be the angle that an element PQ of a curve (Fig. 21) makes with the axis AX, and if $d\vartheta = TQT'$ be the change of direction which takes place in passing from PQ to QR, then, denoting by ds the element on which the change takes place, the curvature of the curve, at that element, will be expressed by $\frac{d\vartheta}{ds}$.

Fig. 21



If a circle be made to pass (as is always possible) through the three points P, Q, R, its curvature will coincide with that of the given curve at the same points; and therefore, designating its radius CQ by R, we shall have

$$ds = Rd\vartheta,$$

and consequently the expression of the curvature will be

$$\frac{d\vartheta}{ds} = \frac{d\vartheta}{Rd\vartheta} = \frac{1}{R}.$$

Now, if x and y be the co-ordinates of the point P, we have

$$\tan \vartheta = \frac{dy}{dx}, \quad \vartheta = \tan^{-1}\left(\frac{dy}{dx}\right), \quad d\vartheta = \frac{\frac{d^2y}{dx}}{1 + \left(\frac{dy}{dx}\right)^2}.$$

On the other hand

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

hence

$$\frac{d\vartheta}{ds} = \frac{\frac{d^2y}{dx}}{1 + \left(\frac{dy}{dx}\right)^2} : \frac{1}{dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{1}{R};$$

and therefore

$$R = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}};$$

that is, the radius of curvature is the radius of the osculatory circle (No. 45); whilst the curvature is the reciprocal of that radius.

When the curve bends upward, $d\vartheta$ is positive, and the value of R is positive; but when the curve bends downward, $d\vartheta$ is negative, and then R is negative.

Remark. We have hitherto assumed that y was the function of an independent variable x; hence dx was constant. If both x and y were functions of a third variable t, then both dx and dy would be variable; and then

$$d\vartheta = \frac{d\frac{dy}{dx}}{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\frac{dxd^2y - dyd^2x}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} = \frac{dxd^2y - dyd^2x}{dx^2 + dy^2};$$

and consequently

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxd^2y - dyd^2x},$$

or, introducing into this value the differential of the new independent variable,

$$R = \frac{\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}.$$

EXAMPLES. *To find the radius of curvature of the parabola.*

The equation $y^2 = 2px$ gives

$$\frac{dy}{dx} = \frac{p}{y}, \qquad \frac{d^2y}{dx^2} = -\frac{p^2}{y^3}.$$

Substituting these values in the expression for R, we find

$$R = -\frac{(y^2 + p^2)^{\frac{3}{2}}}{p^2}.$$

Assume $p = 3$, $y = 4$; then $R = 13.888$ . . .

To find the radius of curvature of the ellipse.

The equation of the ellipse being $a^2y^2 + b^2x^2 = a^2b^2$, we have

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \qquad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3};$$

whence, by substitution and reduction, we find

$$R = -\frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$$

When $y = 0$, we have $x^2 = a^2$, and R reduces to $\frac{b^2}{a}$, which is a minimum. When $x = 0$, then $y^2 = b^2$, and R becomes $= \frac{a^2}{b}$, which is a maximum.

To find the radius of curvature of an equilateral hyperbola referred to its asymptotes.

The equation of the curve being $xy = m^2$, we have

$$\frac{dy}{dx} = -\frac{y}{x}, \qquad \frac{d^2y}{dx^2} = \frac{2y}{x^2};$$

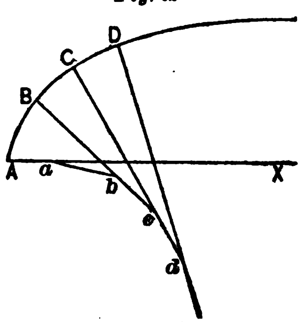
whence, by substitution and reduction, we shall find

$$R = \frac{(m^4 + x^4)^{\frac{3}{2}}}{2m^2x^3}.$$

Evolutes.

47. If we draw osculatory circles to all the points of a curve, and conceive that a line is drawn through the centres of all these circles, this latter line will be the so-called *evolute* of the given curve, and this given curve will then be called its *involute*. Thus *abcd* . . . (Fig. 22) is the evolute of which *ABCD* . . . is the involute.

Fig. 22



The radii of the osculatory circles, being normal to the given curve, have convergent directions, so that each radius produced is intersected by its consecutive. It is evident that the distance between two consecutive points of intersection constitutes an element of the evolute, and that, therefore, the radius of the osculatory circle is at the same time normal to the involute, and tangent to the evolute.

The evolute may be constructed by drawing normals to the curve, and then drawing a curve tangent to all of them; the closer the normals, the more accurate will be the construction.

The involute may be constructed by wrapping a thread around the evolute, holding it tense, and then unwrapping it. Every point of the thread will describe a curve, and every one of these curves will be an involute of the given evolute.

Since the evolute passes through the centres of curvature of the given curve, the co-ordinates of the evolute will be those of these centres, which we have designated (No. 45) by α and β. The equation of the evolute will therefore be found by combining the expressions for α and β with the equation of the given curve, so as to eliminate x, y, and their differentials. We have (No. 45)

$$\alpha = x - \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}} \cdot \frac{dy}{dx}, \quad \beta = y + \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}.$$

If the given curve be the parabola $y^2 = 2px$, we shall have to make $\frac{dy}{dx} = \frac{p}{y}$, and $\frac{d^2y}{dx^2} = -\frac{p^2}{y^3}$; and we shall find

$$\alpha = x + \frac{y^2 + p^2}{p},$$

whence

$$x = \tfrac{1}{3}(\alpha - p),$$

and

$$\beta = y - \frac{y(y^2 + p^2)}{p^2} = -\frac{y^3}{p^2},$$

whence

$$y = -\beta^{\frac{1}{3}} p^{\frac{2}{3}}.$$

These values of x and y being substituted in the equation of the curve, we shall find

$$\beta^{\frac{2}{3}} p^{\frac{4}{3}} = \frac{2p}{3}(\alpha - p),$$

or

$$\beta^2 = \frac{8}{27p}(\alpha - p)^3,$$

which is the equation of the evolute of the parabola.

If the given curve were the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, we would find for its evolute the equation

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

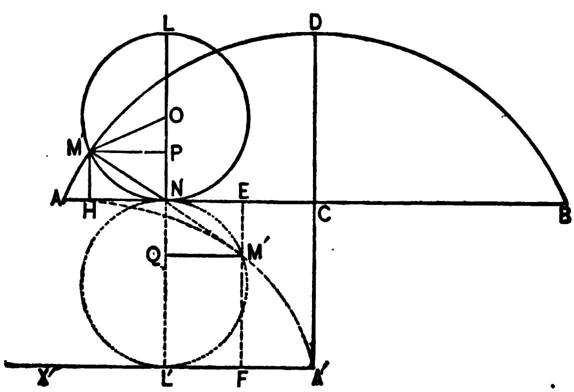
If the given curve were the hyperbola $a^2y^2 - b^2x^2 = -a^2b^2$, we would find for its evolute the equation

$$(a\alpha)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

The length of an arc of the evolute is, in all cases, equal to the difference between the two radii of curvature by which it is intercepted. For, as the elements of the evolute are differentials of consecutive radii, any finite arc of the evolute is a sum of such differentials, and is measured, therefore, by the difference between the intercepting radii.

48. There is an interesting curve, the cycloid,

Fig. 23



whose evolute is another cycloid exactly equal to its involute.

The cycloid is a curve generated by a point in the circumference of a circle, when the circle is rolled along a straight line.

Let AB (Fig. 23) be the straight line, and A the starting point. When the circle has reached the point N, the generating point will have described the portion AM of the curve. Take the origin of co-ordinates at A, and let $AH = x$, $MH = y$, $LN = 2r$. Then

$$AH = AN - NH,$$

or, representing the arc $MN = AN$ by $r\vartheta$,

$$x = r\vartheta - r \sin \vartheta.$$

On the other hand, $y = MH = PN = ON - OP = r - r \cos \vartheta$. Hence

$$\cos \vartheta = \frac{r-y}{r}, \quad \sin \vartheta = \sqrt{1 - \left(\frac{r-y}{r}\right)^2} = \frac{1}{r}\sqrt{2ry - y^2},$$

$$\vartheta = \cos^{-1} \frac{r-y}{r}.$$

Substituting these values in the expression for x, we have

$$x = r \cos^{-1} \frac{r-y}{r} - \sqrt{2ry - y^2},$$

which is the equation of the ascending portion of the cycloid.

From this equation we obtain

$$\frac{dy}{dx} = \frac{\sqrt{2ry - y^2}}{y} = \sqrt{\frac{2r}{y} - 1},$$

whence

$$\frac{dy^2}{dx^2} = \frac{2r}{y} - 1, \text{ and } 2\,\frac{dy}{dx}\,\frac{d^2y}{dx} = -\frac{2rdy}{y^2};$$

$$F(x, y, a) + d_a F(x, y, a) = 0,$$

a alone being supposed to vary. And since we have already the term $F(x, y, a) = 0$, we shall have also $d_a F(x, y, a) = 0$; and the solution of the case will wholly depend on these two equations.

The point of intersection given by these equations changes its position for every different value given to a, and will describe a continuous line when a varies without interruption. If, then, we eliminate a between the same equations, the resulting equation $\varphi(x, y) = 0$ will be the equation of the curve formed by the successive intersection of all the curves derived from the equation $F(x, y, a) = 0$, when a is supposed to vary continuously. The curve $\varphi(x, y) = 0$ will touch all the curves represented by the equation $F(x, y, a) = 0$, and is, therefore, called the *envelope* of these curves.

EXAMPLES. 1. *To find the envelope of a series of parabolas* whose equation is $y^2 = a(x - a)$.

Differentiating the given equation with regard to a, we shall find $a = \frac{1}{2}x$. This value substituted in the equation gives

$$y^2 = \tfrac{1}{4}x^2, \text{ or } y = \pm \tfrac{1}{2}x.$$

The envelope will, therefore, consist of two straight lines.

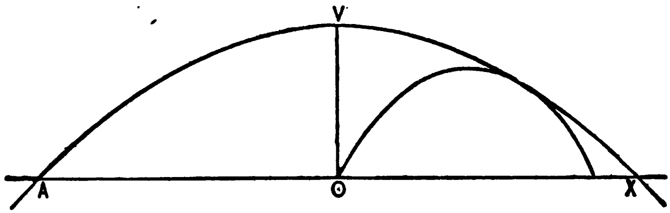
2. *A projectile is discharged from the point* O (Fig. 24) *with a constant velocity, but at different inclinations to the horizon. To find the envelope of the curves it describes.*

We know by mechanics that the path of a projectile is a parabola represented by the equation

$$y = ax - (1 + a^2)\frac{x^2}{4h},$$

in which x and y are the co-ordinates of the curve, a the trigonometric tangent of the angle of projection, and $h = OV$ the height due to the velocity of

Fig. 24



projection. Differentiating this equation with regard to a, we shall find $a = \frac{2h}{x}$; and this value substituted in the equation will give

$$x^2 = 4h(h - y)$$

for the equation of the envelope. This envelope is a parabola AVX having its focus at the point O and its vertex at V, its parameter being $= 4h$.

3. *Given the area of a right-angled triangle, to find the curve to which the hypotenuse is always a tangent.*

Let the given area be $2m^2$, and let $AM = x$, $MP = y$ (Fig. 25) be the co-ordinates of a point P common to the hypotenuse and to the curve. If $AT = a$ be the parameter by the variation of which the triangle changes its position, we shall have

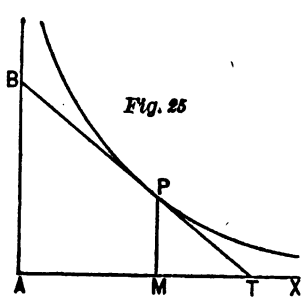


Fig. 25

$$AB = \frac{4m^2}{a}, \ y = (a - x)\tan ATB = (a - x)\frac{4m^2}{a^2},$$

whence

$$a^2 y = 4m^2 (a - x).$$

Differentiating this equation with regard to a, we find $a = \frac{2m^2}{y}$; and this value substituted in our equation gives

$$xy = m^2;$$

and therefore the curve is a hyperbola referred to its asymptotes.

50. The above theory may also be applied to the solution of problems in which the parameter depends on the variation of two quantities. In this case, the data of the problem must give rise to two equations. Let us give an example.

The straight line PQ (Fig. 26) *slides between two rectangular axes. Find the envelope, or the locus of its intersections.*

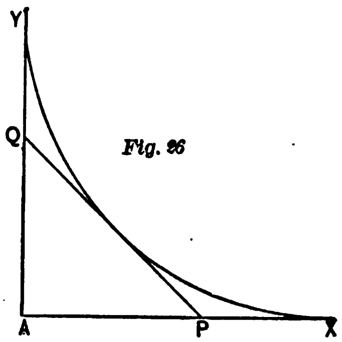


Fig. 26

For the solution, let AX, AY be the co-ordinate axes. Let $AP = a$, $AQ = b$, $PQ = c$. Then the equation of the line PQ will be

$$y = -\frac{b}{a}x + b,$$

which, for greater convenience, may be written thus:

$$\frac{x}{a}+\frac{y}{b}=1, \qquad (1)$$

and in which a and b are subject to the condition

$$a^2+b^2=c^2. \qquad (2)$$

Differentiating (1) and (2) with regard to a and b, we obtain

$$\frac{xda}{a^2}+\frac{ydb}{b^2}=0, \quad ada+bdb=0,$$

whence, by eliminating da and db, we shall find

$$\frac{x}{a^3}=\frac{y}{b^3}, \text{ or } \frac{x}{a}=\frac{a^2y}{b^3}, \text{ and } \frac{y}{b}=\frac{b^2x}{a^3}.$$

Substituting, in succession, these values in (1), we obtain

$$y(a^2+b^2)=b^3 \text{ and } x(a^2+b^2)=a^3,$$

or

$$yc^2=b^3, \text{ and } xc^2=a^3;$$

whence

$$a=x^{\frac{1}{3}}c^{\frac{2}{3}},\ b=y^{\frac{1}{3}}c^{\frac{2}{3}}.$$

Substituting these values of a and b in equation (1), and reducing, we finally obtain the equation of the envelope, which is

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}.$$

Elements of Arcs, Surfaces, and Volumes.

51. We have seen (No. 33) that if ds is an infinitesimal element of the line s referred to rectangular axes, we have

$$ds=\sqrt{dx^2+dy^2},$$

the values of dx and dy being drawn from the equation of the line. This equation gives *the differential of any arc.*

The differential of an area is the infinitesimal area comprised between two consecutive ordinates of the curve. Thus, if AP and BQ (Fig. 27) are two consecutive ordinates, the area $APQB$ will be the differential of the area comprised between the curve and the axis OX. Now $APQB = ydx + \frac{1}{2}dydx$. Hence, denoting $APQB$ by dA, and neglecting the term $\frac{1}{2}dydx$, we shall have

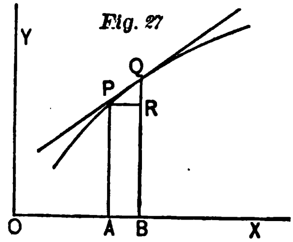


Fig. 27

$$dA = ydx$$

as the expression for *the differential of a plane area.*

If the element PQ of the curve be made to revolve about the axis OX, it will generate an infinitesimal element of a conical surface. Denote it by dS. Its expression will be

$$dS = \tfrac{1}{2}\{2\pi y + 2\pi(y + dy)\}\sqrt{dx^2 + dy^2},$$

or, reducing,

$$dS = 2\pi y\sqrt{dx^2 + dy^2};$$

and this is *the differential of a surface of revolution.*

If the area $APQB$ be made to revolve about the axis OX, it will generate an infinitesimal element of a conical volume. Call it dV. Then

$$dV = \tfrac{1}{3}\{\pi y^2 + \pi(y + dy)^2 + \pi y(y + dy)\}dx,$$

or, reducing,

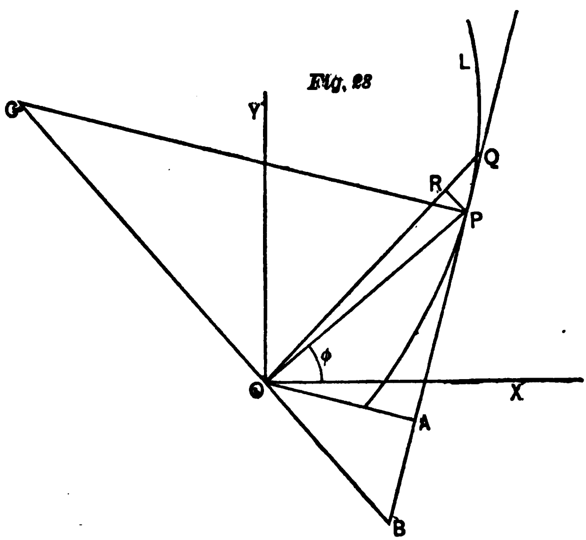
$$dV = \pi y^2 dx;$$

and this is *the differential of a volume of revolution.*

Differentials with Polar Co-ordinates.

52. Let $\rho = f(\varphi)$ be the polar equation of any curve PL (Fig. 28), the pole being at O, and OX being the initial line.

At the point P we have $\rho = OP$, and $\varphi = POX$.



Draw the tangent PB, the normal PC, and through the pole O draw BC perpendicular to the radius vector. Then OB will be the *subtangent*, and OC the *subnormal*, while OA perpendicular to the tangent will be the *polar distance of the tangent.*

Let $ds = PQ$ be an infinitesimal element of the curve. Draw the radius vector OQ, and, with O as centre, the arc PR. The infinitesimal triangle PQR may be considered rectilinear, and right-angled at R. Hence

$$\overline{PQ}^2 = \overline{PR}^2 + \overline{RQ}^2.$$

But $PQ = ds$, $PR = \rho d\varphi$, $RQ = d\rho$; hence, substituting and extracting the root,

$$ds = \sqrt{d\rho^2 + \rho^2 d\varphi^2}.$$

Such is *the differential of the arc.*

Let dA be the differential of the area swept over by the radius vector. The infinitesimal area POQ may be considered as the area of a rectilinear triangle having the base OQ and the altitude PR. Hence

$$dA = \tfrac{1}{2} OQ \times PR = \tfrac{1}{2}(\rho + d\rho)\rho d\varphi,$$

or, as $d\rho$ disappears by the side of ρ,

$$dA = \tfrac{1}{2}\rho^2 d\varphi.$$

Such is *the differential of the area.*

Let V denote the angle OPB of the tangent with the radius vector. This angle and the angle RQP, may be considered equal, as they differ only infinitesimally. Now

$$\tan RQP = \frac{PR}{QR} = \frac{\rho d\varphi}{d\rho};$$

therefore

$$\tan V = \frac{\rho d\varphi}{d\rho}.$$

Let $OA = p$. We have $OA = OP \sin OPA$; therefore

$$p = \rho \sin V, \quad \text{or} \quad p = \rho \frac{\tan V}{\sqrt{1 + \tan^2 V}}.$$

Substituting the value of tan V, and reducing, we have

$$p = \frac{\rho^2 d\varphi}{\sqrt{d\rho^2 + \rho^2 d\varphi^2}}.$$

Such is *the polar distance of the tangent.*

Let $S.T$ denote the subtangent. We have

$$S.\,T = OB = OP.\tan V,$$

or

$$S.\,T = \rho^2 \frac{d\varphi}{d\rho}.$$

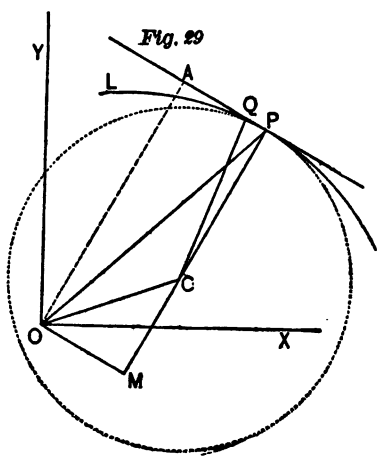
Let $S.N$ denote the subnormal. The proportion

$$OC : OP :: OP : OB$$

gives

$$S.N = OC = \frac{\overline{OP}^2}{OB} = \frac{\rho^2}{\rho^2 \frac{d\varphi}{d\rho}} = \frac{d\rho}{d\varphi}.$$

Let $PC = R$ (Fig. 29) be the radius of curvature of the line PL at the point P. Drawing OM perpendicular to PC produced, and joining the points O and C, we have from the triangle POC



$$\overline{OC}^2 = \overline{OP}^2 + \overline{PC}^2 - 2\,OP\,.\,PC\cos OPC.$$

But $OP \cos OPC = PM = OA = p$.

Therefore

$$\overline{OC}^2 = \rho^2 + R^2 - 2pR.$$

Now, if we pass from P to Q, the side OC and the radius R will remain unchanged, ρ and p alone being subject to variation. Differentiating, then, and considering OC and R as constants, we find

$$o = 2\rho d\rho - 2Rdp, \text{ whence } R = \frac{\rho d\rho}{dp}.$$

Spirals.

53. A spiral is a plane curve generated by a point moving on a straight line while this straight line is uniformly revolving about a fixed point or *pole.* The portion of the curve generated during one revolution is called a *spire.* The law according to which the moving point advances along the revolving line determines the nature of the spiral. Denoting by ρ the radius vector, and by φ the angle that the radius vector makes with the initial line, and considering ρ as a function of φ, the general equation of a spiral will be

$$\rho = f(\varphi).$$

The most remarkable spirals are the *spiral of Archimedes*, the *parabolic spiral*, the *hyperbolic spiral*, and the *logarithmic spiral.*

54. The *spiral of Archimedes* is generated by a point moving uniformly along a straight line uniformly revolving. If ρ and ρ' be two radii vectores, and φ and φ' the corresponding angles, by the law of uniform variation we shall have

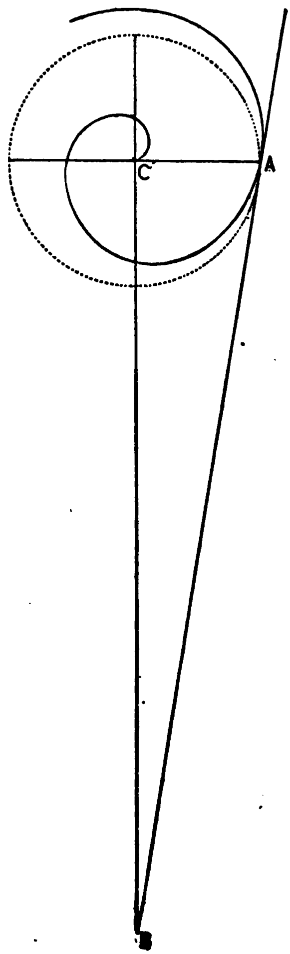
$$\rho : \rho' :: \varphi : \varphi', \text{ whence } \rho = \rho' \frac{\varphi}{\varphi'}.$$

Let us take $AC = r$ as the radius of a measuring circle (Fig. 30), and let $\rho' = r$ correspond to $\varphi' = 2\pi$.

Then $\rho = \frac{r\varphi}{2\pi}$, or, making for greater simplicity $\frac{r}{2\pi} = a$,

$$\rho = a\varphi. \qquad (1)$$

Fig. 30



Such is the equation of the spiral of Archimedes. Its form is the same as that of the equation $y = ax$ of a straight line passing through the origin of co-ordinates.

This spiral may be constructed by dividing the circumference into a number of equal parts, and the radius into the same number of equal parts, and then taking as many parts of r for the radius vector as there are corresponding parts taken on the circumference.

Differentiating (1) we have $d\rho = ad\varphi$ Substituting this in the general formulas of No. 52, we shall find

$$ds = ad\varphi \sqrt{1 + \varphi^2},$$
$$dA = \tfrac{1}{2}a^2\varphi^2 d\varphi,$$
$$\tan V = \frac{\rho}{a} = \varphi,$$
$$p = \frac{a\varphi^2}{\sqrt{1 + \varphi^2}},$$
$$R = \frac{a\sqrt{(1 + \varphi^2)^3}}{2 + \varphi^2},$$
$$S.T = a\varphi^2, \quad S.N = a.$$

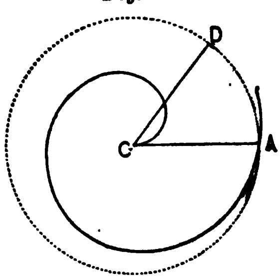
It will be remarked that the subtangent CB corresponding to $\varphi = 2\pi$ is equal to the circumference of the measuring circle; for $a\varphi^2$ becomes $a(2\pi)^2 = \frac{r}{2\pi}(2\pi)^2 = 2\pi r$. Hence the area of the triangle ABC is $\frac{1}{2}r \times 2\pi r = \pi r^2 =$ area of the circle.

55. The *parabolic spiral* (Fig. 31) is so called because its equation

$$\rho^2 = 2a\varphi \qquad (2)$$

is of the same form as the equation $y^2 = 2ax$ of the parabola. It may be constructed by first constructing the parabola and then laying off from A to D along the circumference of a measuring circle any assumed abscissa x, and drawing from the centre C towards D the corresponding ordinate y as a radius vector. Let r be the radius of the measuring circle; then $\frac{x}{r}$ becomes φ, and y becomes ρ. Hence after a revolution we shall have $\frac{x}{r} = 2\pi$, and $y = CA = r$, and the equation (2) will give $r^2 = 2a \times 2\pi$, or $a = \frac{r^2}{4\pi}$.

Fig. 31



Differentiating (2), we have $\rho d\rho = a d\varphi$. Substituting this in the general formulas of No. 52, we shall find

$$ds = d\varphi\sqrt{\frac{a(1+4\varphi^2)}{2\varphi}},\ dA = a\varphi d\varphi,\ \tan V = \frac{2a\varphi}{a} = 2\varphi,$$

$$p = 2\varphi\sqrt{\frac{2a\varphi}{1+4\varphi^2}}, \quad R = \frac{a(1+4\varphi^2)^2}{(3+4\varphi^2)\sqrt{2a\varphi(1+4\varphi^2)}},$$

$$S.T = 2\varphi\sqrt{2a\varphi}, \qquad S.N = \frac{a}{2a\varphi} = \frac{1}{2\varphi}.$$

56. The *hyperbolic spiral* is so called because its equation

$$\rho\varphi = a \tag{3}$$

is of the same form as the equation $xy = m^2$ of the hyperbola referred to its centre and asymptotes. From (3) we obtain

$$\rho : \rho' :: \frac{1}{\varphi} : \frac{1}{\varphi'};$$

hence the radii vectores are inversely proportional to the varying angles.

Differentiating (3), we have $d\rho = -\frac{ad\varphi}{\varphi^2}$. Substituting this in the general formulas of No. 52, we shall find

$$ds = \frac{ad\varphi}{\varphi^2}\sqrt{1+\varphi^2}, \ dA = \frac{a^2 d\varphi}{2\varphi^2}, \ \tan V = -\varphi,$$

$$p = \frac{a}{\sqrt{1+\varphi^2}}, \ R = \frac{a}{\varphi^4}\sqrt{(1+\varphi^2)^3},$$

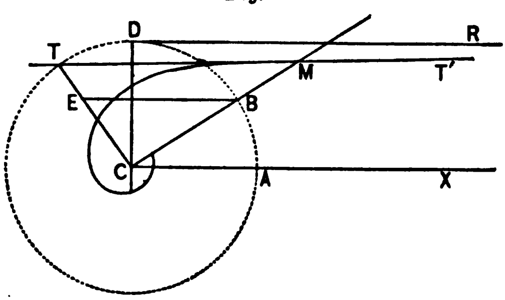
$$S.T = -a, \ S.N = -\frac{a}{\varphi^2}.$$

The equation of this spiral shows that when $\varphi = 0$, then $\rho = \infty$; which means that the curve begins at an infinite distance from the pole. On the other hand, ρ cannot become $= 0$ unless φ becomes infinite; and this shows that the curve can make an infinite number of spires in approaching the pole.

Again, when $\varphi = 0$, the curve and its tangent are parallel to the initial line CX, and the equation of the polar distance of the tangent gives $p = a$. Erecting at C (Fig. 32) $CD = a$, perpendicular to CX, and drawing DR parallel to CX, the line DR, if prolonged to infinity, would be the tangent in question.

The equation $\tan V = -\varphi$ suggests the following manner of constructing the curve by points. With

Fig. 32



C as centre and $CD = a$ as radius, describe a circle. Draw the indefinite line CM and the radius CT perpendicular to it. The arc AB, which measures the angle $ACB = \varphi$, has a length $a\varphi$. On CT take $CE = a\varphi = AB$, and draw EB; then from T draw TM parallel to EB. The point M will be a point of the curve. For, by our construction, we have

$$EC = a \tan CBE = a \tan CMT = a\varphi = AB,$$

hence

$$\tan CMT = \varphi.$$

But $\tan CMT = -\tan CMT'$; therefore

$$-\tan CMT' = \varphi, \text{ or } \tan CMT' = -\varphi = \tan V;$$

hence the point M belongs to the curve. Other points may be determined by the same method.

57. The *logarithmic spiral* is represented by the equation

$$\varphi = a \log \rho, \tag{4}$$

in which φ is considered as a function of ρ.

Differentiating (4), we obtain $d\varphi = \frac{a d\rho}{\rho}$. Substituting this in the general formulas of No. 52, we shall find

$$ds = d\rho \sqrt{1 + a^2}; \; dA = \frac{a\rho d\rho}{2}, \; \tan V = a,$$

$$p = \frac{a\rho}{\sqrt{1 + a^2}}, \; R = \frac{\rho}{a}\sqrt{1 + a^2}, \; S.T = a\rho, \; S.N = \frac{\rho}{a}.$$

We see that the angle formed by the radius vector and the tangent is constant, and that the trigonometric tangent of this angle is equal to the modulus a of the logarithms used. In the Napierian system we have $a = 1$, and consequently $V = 45°$.

Since $\frac{\varphi}{a}$ is the Napierian logarithm of ρ, the arithmetical progression

$$0, \frac{\varphi}{a}, \frac{2\varphi}{a}, \frac{3\varphi}{a}, \ldots$$

will entail a geometric progression

$$1, e^{\frac{\phi}{a}}, e^{\frac{2\phi}{a}}, e^{\frac{3\phi}{a}}, \ldots$$

for the corresponding radii vectores. Hence, if we describe a circle with the radius $\frac{1}{a}$, and divide its circumference into equal parts, and if then from its centre we draw through each point of division indefinite straight lines, and on these lines we take such lengths as are required in order to form a series of radii vectores in geometrical progression $\left(\text{the ratio being } e^{\frac{\phi}{a}}\right)$, we shall determine any number of points belonging to the spiral.

PART II.

INTEGRAL CALCULUS.

58. The object of the integral calculus is to explain how to pass from given differentials to the functions from which they may be derived by differentiation. The functions thus obtained are called *integrals*, and the operation by which they are found is called *integration.*

To express that a function x is the integral of dx, we write

$$\int dx = x,$$

and the sign $\int$ prefixed to the differential is called the *integral sign.*

Integration and differentiation are inverse operations. It follows, first, that, since the differential of a polynomial is the algebraic sum of the differentials of its terms, the integral of a differential polynomial must be the algebraic sum of the integrals of all its terms.

Secondly, since *constant factors* are not subject to differentiation, they are not subject to integration, and may, therefore, be written without the sign $\int$; thus, if a is a constant,

$$\int adx = a\int dx = ax.$$

Thirdly, since a *constant term* disappears by differentiation, a constant must be added to the integral obtained, to make it *complete.* We determine the value of such a constant, in each particular case, by making it agree with the conditions of the problem proposed; but, as it is capable of being determined by various arbitrary conditions, it is commonly considered as an *arbitrary constant.*

Before any value has been assigned to the constant the integral is said to be *indefinite;* when the value of the constant has been determined so as to satisfy a particular hypothesis, the integral is said to be *particular;* and when, moreover, a definite value has been assigned to the variable, the integral is said to be *definite.*

59. The following example will show how definite integrals can be obtained. Let $3x^2dx$ be the function to be integrated. We already know that $d(x^3) = 3x^2dx$. We have therefore

$$\int 3x^2dx = x^3;$$

and this is the *incomplete* integral. Add to it a constant C; then

$$\int 3x^2dx = x^3 + C;$$

and this is a *complete* integral; but it is still *indefinite.*

Assume now that the particular problem under consideration requires the integral to be $=0$ when $x=a$, that is, that our integral begins at $x=a$. In this case we shall have

$$0 = a^3 + C,$$

and consequently $C = -a^3$. Hence we shall write

$$\int_a^x 3x^2dx = x^3 - a^3,$$

the letter a placed at the bottom of the sign $\int$ indicating the *inferior* limit of the integral, viz., the place where it begins; and the letter x at the top of the same sign indicating the *superior* limit where the integral ends. But, as x is still variable, this is a *particular* integral. If our problem requires the integral to end at $x = b$, we shall have at last

$$\int_a^b 3x^2 dx = b^3 - a^3,$$

and this will be the *definite* integral.

In general, if we have

$$\int f(x)\,dx = F(x) + C,$$

assigning to the integral the limits required by the conditions of the case, say a and b, the *definite* integral will be expressed by

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The present treatise will contain three sections. In the first we shall explain the various methods of integration; in the second we shall apply the integral calculus to many questions of Geometry; and in the third we shall solve by it various mechanical problems.

SECTION I.

VARIOUS METHODS FOR FINDING INTEGRALS.

60. Before we proceed further, let the student be warned that, although in the differential calculus we can always pass, by a uniform method, from a given function to its differential, in the integral calculus we have no general method for passing in all cases from a given differential to its integral; and for this reason we are not unfrequently compelled to make use of devices of various kinds, the choice and employment of which is not always exempt from difficulties.

We shall start from the fundamental formulas which regard the simplest cases of integration; after which we shall explain various processes used for the reduction of other less simple cases to forms integrable by the same fundamental formulas.

Integration of Elementary Forms.

61. The fundamental formulas of the integral calculus are found without labor by simply reversing the corresponding formulas obtained in the differential calculus.

Thus, since the expression $\frac{ax^{n+1}}{n+1}$ has $ax^n dx$ for its differential, we immediately see that the integral of $ax^n dx$ is $\frac{ax^{n+1}}{n+1}$. Therefore

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C. \qquad (1)$$

This formula is general, whether n be positive or negative, integer or fractional. The only case to be excepted is $n = -1$, because in this case $\frac{ax^{n+1}}{n+1}$ becomes $\frac{a}{0}$, and gives no differential.

On the other hand, $ax^{-1}dx$ is not the differential of a power, but of a logarithmic function; for $ax^{-1}dx = a\frac{dx}{x} = ad \log x$; and thus

$$\int ax^{-1}dx = a\int\frac{dx}{x} = a \log x + C. \qquad (2)$$

The expression $a^x \log a dx$ is the differential of a^x; hence

$$\int a^x dx = \frac{a^x}{\log a} + C. \qquad (3)$$

By reversing the differential trigonometric formulas of No. 17, we shall find the following :

$$\int \cos x dx = \sin x + C. \qquad (4)$$

$$\int \sin x dx = -\cos x + C. \qquad (5)$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C. \qquad (6)$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C. \qquad (7)$$

$$\int \sin x dx = \text{vers } x + C. \qquad (8)$$

$$\int \cos x dx = -\text{covers } x + C. \qquad (9)$$

$$\int \frac{\sin x dx}{\cos^2 x} = \sec x + C. \qquad (10)$$

$$\int \frac{\cos x dx}{\sin^2 x} = -\text{cosec } x + C. \qquad (11)$$

In like manner, if we reverse the differential formulas of No. 18, we shall find the following:

$$\int \frac{dy}{\sqrt{1-y^2}} = \sin^{-1} y + C. \qquad (12)$$

$$\int \frac{dy}{\sqrt{1-y^2}} = -\cos^{-1} y + C. \qquad (13)$$

$$\int \frac{dy}{1+y^2} = \tan^{-1} y + C. \qquad (14)$$

$$\int \frac{dy}{1+y^2} = -\cot^{-1} y + C. \qquad (15)$$

$$\int \frac{dy}{\sqrt{2y-y^2}} = \text{vers}^{-1} y + C. \qquad (16)$$

$$\int \frac{dy}{\sqrt{2y-y^2}} = -\text{covers}^{-1} y + C. \qquad (17)$$

$$\int \frac{dy}{y\sqrt{y^2-1}} = \sec^{-1} y + C. \qquad (18)$$

$$\int \frac{dy}{y\sqrt{y^2-1}} = -\text{cosec}^{-1} y + C. \qquad (19)$$

In these latter formulas, make $y = \frac{bx}{a}$; then

$dy = \frac{b}{a}\,dx$; and by substitution and reduction we shall obtain the following:

$$\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b}\sin^{-1}\frac{bx}{a} + C. \tag{20}$$

$$\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = -\frac{1}{b}\cos^{-1}\frac{bx}{a} + C. \tag{21}$$

$$\int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab}\tan^{-1}\frac{bx}{a} + C. \tag{22}$$

$$\int \frac{dx}{a^2 + b^2x^2} = -\frac{1}{ab}\cot^{-1}\frac{bx}{a} + C. \tag{23}$$

$$\int \frac{dx}{\sqrt{2abx - b^2x^2}} = \frac{1}{b}\operatorname{vers}^{-1}\frac{bx}{a} + C. \tag{24}$$

$$\int \frac{dx}{\sqrt{2abx - b^2x^2}} = -\frac{1}{b}\operatorname{covers}^{-1}\frac{bx}{a} + C. \tag{25}$$

$$\int \frac{dx}{x\sqrt{b^2x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{bx}{a} + C. \tag{26}$$

$$\int \frac{dx}{x\sqrt{b^2x^2 - a^2}} = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{bx}{a} + C. \tag{27}$$

By these formulas a great number of differentials can be integrated. We give here a few examples for the exercise of the student.

EXAMPLES.

1. $dy = x\,dx$, $\qquad y = \frac{1}{2}x^2$.
2. $dy = \frac{a\,dx}{\sqrt{x}}$, $\qquad y = 2a\sqrt{x}$.
3. $dy = ax^3dx$, $\qquad y = \frac{1}{4}ax^4$.

4. $dy = 2x^{\frac{5}{2}}dx,$ $\qquad y = \frac{4}{7}x^{\frac{7}{2}},$

5. $dy = -\frac{2}{3}x^{-\frac{6}{5}}dx,$ $\qquad y = \frac{10}{3}x^{-\frac{1}{5}},$

6. $dy = x^{-\frac{m}{n}}\,dx,$ $\qquad y = \dfrac{n}{n-m}\,x^{\frac{n-m}{n}}.$

7. $dy = 2x^{-1}\,dx,$ $\qquad y = \log(x^2).$

8. $dy = \left(2x - \dfrac{1}{x}\right)dx,$ $\qquad y = x^2 - \log x.$

9. $dy = a^{\sin x}\cos x\,dx,$ $\qquad y = \dfrac{a^{\sin x}}{\log a}.$

10. $dy = 2\cos(2x)\,dx,$ $\qquad y = \sin 2x.$

11. $dy = \dfrac{dx}{\cos^2\left(\dfrac{x}{2}\right)},$ $\qquad y = 2\tan\dfrac{x}{2}.$

12. $dy = \sin(ax)\,dx,$ $\qquad y = \dfrac{1}{a}\operatorname{vers}(ax).$

13. $dy = x\cos\left(\dfrac{x^2}{2}\right)dx,$ $\qquad y = -\operatorname{covers}\dfrac{x^2}{2}.$

14. $dy = \dfrac{dx}{\sqrt{1-4x^2}},$ $\qquad y = \frac{1}{2}\sin^{-1}2x.$

15. $dy = -\dfrac{2x\,dx}{\sqrt{1-x^4}},$ $\qquad y = \cos^{-1}(x^2).$

16. $dy = \dfrac{x\,dx}{1+x^4},$ $\qquad y = \frac{1}{2}\tan^{-1}(x^2).$

17. $dy = \frac{dx}{\sqrt{6x - 9x^2}}$, $y = \frac{1}{3} \text{vers}^{-1} 3x.$

18. $dy = \frac{2xdx}{3\sqrt{2 - 5x^4}}$, $y = \frac{1}{3\sqrt{5}} \sin^{-1}\left(\frac{x^2\sqrt{5}}{\sqrt{2}}\right).$

19. $dy = -\frac{2dx}{4 + x^2}$, $y = \cot^{-1}\frac{x}{2}.$

20. $dy = \frac{2dx}{x\sqrt{3x^2 - 5}}$, $y = \frac{2}{\sqrt{5}} \sec^{-1}\left(\frac{x\sqrt{3}}{\sqrt{5}}\right).$

21. $dy = \frac{\sin x dx}{\sqrt{1 - 4\cos^2 x}}$, $y = \frac{1}{2}\cos^{-1}(2\cos x).$

22. $dy = \frac{\cos x dx}{1 + \sin^2 x}$, $y = \tan^{-1}(\sin x).$

23. $dy = \frac{a^x dx}{1 + a^{2x}}$, $y = \frac{1}{\log a} \tan^{-1} a^x.$

In the above examples we have omitted the arbitrary constant, as it had no bearing on the work proposed.

62. When the fundamental formulas above given are used for the successive integration of differentials of the second, third, or any higher order, a constant is to be added at each integration.

Thus, if the expression $\frac{d^3y}{dx^3} = ax$ is to be integrated, we first multiply both its members by dx, which gives us

$$\frac{d^3y}{dx^2} = axdx, \text{ or rather } d\,\frac{d^2y}{dx^2} = axdx.$$

Integrating this, and adding a constant, we have

$$\frac{d^2y}{dx^2} = \frac{ax^2}{2} + C.$$

Multiplying this first integral by dx, we have

$$d\left(\frac{dy}{dx}\right) = \frac{ax^2}{2}\,dx + C dx,$$

and integrating, and adding a new constant,

$$\frac{dy}{dx} = \frac{ax^3}{6} + Cx + C'.$$

Multiplying this second integral by dx, integrating, and adding a third constant, we shall at last obtain

$$y = \frac{ax^4}{24} + \frac{Cx^2}{2} + C'x + C''.$$

It is a general rule that every complete integral must contain as many arbitrary constants as there have been successive integrations performed for obtaining it. Each constant is, of course, to be determined by taking the integral between the limits required by the particular conditions of the problem to be solved.

Reduction of Differentials to an Elementary Form.

63. When the proposed differential is not in the form required for integration, it may often be reduced to a proper form by some simple algebraic process. Thus the differential

$$(a^2 + x^2)^2\,dx$$

will be reduced to an integrable form *by performing the operation* indicated by the exponent of the parenthesis. We shall then have

$$\int (a^2 + x^2)^2 \, dx =$$

$$\int (a^4 + 2a^2x^2 + x^4) \, dx = a^4x + \frac{2a^2}{3} x^3 + \tfrac{1}{5}x^5.$$

Thus also the differential $\frac{x^3dx}{a - x}$ will be properly reduced *by division.* As

$$\frac{x^3dx}{a - x} = -\left(x^2 + ax + a^2 - \frac{a^3}{a - x}\right) dx,$$

hence

$$\int \frac{x^3dx}{a - x} = -\frac{x^3}{3} - \frac{ax^2}{2} - a^2x - a^3 \log (a - x).$$

If we had a differential of the form

$$\frac{(a + bx) \, dx}{a^2 + x^2},$$

we might *split it* into its parts by writing

$$\frac{adx}{a^2 + x^2} + \frac{bxdx}{a^2 + x^2},$$

whence

$$\int \frac{(a + bx) \, dx}{a^2 + x^2} = \tan^{-1} \frac{x}{a} + b \log \sqrt{a^2 + x^2}.$$

In like manner the differential

$$\frac{xdx}{\sqrt{2ax - x^2}}$$

will be reduced to two integrable terms by simply adding and subtracting the quantity a in its numerator. We obtain thus

$$\frac{xdx}{\sqrt{2ax-x^2}} = \frac{adx}{\sqrt{2ax-x^2}} - \frac{(a-x)\,dx}{\sqrt{2ax-x^2}};$$

whence

$$\int \frac{xdx}{\sqrt{2ax-x^2}} = a \operatorname{vers}^{-1}\frac{x}{a} - \sqrt{2ax-x^2}.$$

The differential

$$\frac{dx\sqrt{x^2-a^2}}{x}$$

may be reduced to an integrable form by multiplying its numerator and its denominator by $\sqrt{x^2-a^2}$. We thus obtain

$$\frac{dx\sqrt{x^2-a^2}}{x} =$$

$$\frac{x^2-a^2}{x\sqrt{x^2-a^2}}\,dx = \frac{xdx}{\sqrt{x^2-a^2}} - \frac{a^2dx}{x\sqrt{x^2-a^2}};$$

whence

$$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a}.$$

64. Sometimes a differential is reduced to an integrable form by the introduction of an *auxiliary variable.* Let

$$dy = x^2\sqrt{a+x}dx.$$

If we make $a+x=z$, whence $dx=dz$, $x^2=(z-a)^2$, we obtain

$$dy = (z-a)^2 z^{\frac{1}{2}}\,dz,$$

which, being developed, gives

$$dy = (z^{\frac{5}{2}} - 2az^{\frac{3}{2}} + a^2 z^{\frac{1}{2}})\,dz,$$

and

$$y = \tfrac{2}{7} z^{\frac{7}{2}} - \tfrac{4}{5}\, az^{\frac{5}{2}} + \tfrac{2}{3}\, a^2 z^{\frac{3}{2}},$$

or, replacing z by $a + x$, and reducing,

$$y = \left\{ \tfrac{2}{7}(a + x)^2 - \frac{4a}{5}(a + x) + \frac{2a^2}{3} \right\} \sqrt{(a + x)^3}.$$

Again, let it be required to integrate the differential

$$dy = \frac{dx}{\sqrt{x^2 \pm a^2}}.$$

Assume $x^2 \pm a^2 = z^2$, whence $xdx = zdz$. Adding zdx to both members of this last equation, and factoring, we have

$$(x + z)\,dx = z\,(dx + dz),$$

whence

$$\frac{dx}{z} = \frac{dx + dz}{x + z} = \frac{d\,(x + z)}{x + z}.$$

Therefore

$$y = \int \frac{dx}{z} = \int \frac{d\,(x + z)}{x + z} = \log\,(x + z);$$

that is,

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\,(x + \sqrt{x^2 \pm a^2}), \qquad (28)$$

a formula of great use in calculus.

If we had

$$dy = \frac{dx}{\sqrt{2ax + x^2}},$$

we might reduce the denominator to the form $\sqrt{(a+x)^2-a^2}$, and the numerator dx to the form $d(a+x)$; whence by formula (28),

$$\int\frac{dx}{\sqrt{2ax+x^2}}=$$

$$\int\frac{d(a+x)}{\sqrt{(a+x)^2-a^2}}=\log(a+x+\sqrt{2ax+x^2}), \tag{29}$$

which is another very useful formula.

Integration by Parts.

65. We have found (No. 12) that the differential of the product uv of two variables is

$$d(uv)=udv+vdu;$$

and therefore

$$uv=\int udv+\int vdu,$$

whence

$$\int udv=uv-\int vdu. \tag{30}$$

By this formula we are enabled to integrate the expression udv whenever we can integrate the expression vdu. This method of integration is called *integration by parts.*

As a first example, let

$$dy=x^3dx\sqrt{a^2-x^2}.$$

To integrate this expression, make

$$x^2=u, \text{ and } xdx\sqrt{a^2-x^2}=dv;$$

then

$$du=2xdx, \text{ and } v=-\tfrac{1}{3}\sqrt{(a^2-x^2)^3}.$$

Substituting these values in formula (30), we have

$$\int x^3 dx \sqrt{a^2 - x^2} =$$

$$- \frac{x^2}{3} \sqrt{(a^2 - x^2)^3} + \int \frac{2x dx}{3} \sqrt{(a^2 - x^2)^3},$$

and, by integrating the last term of this equation,

$$y = \int x^3 dx \sqrt{a^2 - x^2} = - \frac{x^2}{3} \sqrt{(a^2 - x^2)^3} - \tfrac{2}{15} \sqrt{(a^2 - x^2)^5}.$$

As a second example, let

$$dy = x \log x dx.$$

Make $\log x = u$, and $x dx = dv$. Then $du = \frac{dx}{x}$, and $v = \frac{x^2}{2}$; and formula (30) will give

$$\int x \log x dx = \frac{x^2}{2} \log x - \int \frac{x dx}{2},$$

that is,

$$y = \int x \log x dx = \frac{x^2}{2} \log x - \frac{x^2}{4}.$$

As a third example, let

$$dy = dx \sqrt{a^2 - x^2}.$$

Make $\sqrt{a^2 - x^2} = u$, $dx = dv$. Then $v = x$, and $du = - \frac{x dx}{\sqrt{a^2 - x^2}}$; and formula (30) will give

$$\int dx \sqrt{a^2 - x^2} = x \sqrt{a^2 - x^2} + \int \frac{x^2 dx}{\sqrt{a^2 - x^2}}.$$

On the other hand,

$$\int dx\,\sqrt{a^2-x^2} =$$

$$\int dx\,\frac{a^2-x^2}{\sqrt{a^2-x^2}} = \int\frac{a^2dx}{\sqrt{a^2-x^2}} - \int\frac{x^2dx}{\sqrt{a^2-x^2}};$$

hence, adding together these two equations,

$$2\int dx\,\sqrt{a^2-x^2} = x\sqrt{a^2-x^2} + \int\frac{a^2dx}{\sqrt{a^2-x^2}},$$

whence

$$\int dx\,\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

Integration of Rational Fractions.

66. When a fraction whose terms are rational and entire is to be integrated, the variable must have a lower degree in the numerator than in the denominator. When the degree is too high, it is reduced by division. The quotient will then consist of an entire rational part, plus a remainder divided by the denominator. Thus

$$\frac{x^4dx}{a^2+x^2} = \left(x^2 - a^2 + \frac{a^4}{a^2+x^2}\right)dx.$$

As there will be no difficulty in integrating the entire part of the quotient, we have only to show here how any remaining fractional part can be integrated.

There are four cases. The first regards a fraction whose denominator has *real unequal* roots, and can be decomposed into *unequal* binomial factors of the first degree. The second regards a fraction whose denominator has some *real equal* roots, and contains some *equal* binomial factors of the first degree. The third regards a fraction whose denominator has *unequal imaginary* roots. The fourth regards a fraction whose denominator has some *equal imaginary* roots.

67. *First case: All roots real and unequal.* Let

$$dy = \frac{(2x-5)\,dx}{(x-1)(x-2)(x-3)}.$$

Assume

$$\frac{2x-5}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3},$$

where A, B, and C are quantities to be determined. Clearing of fractions, we have

$$2x-5 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Performing the operations indicated, and equating the coefficients of the like powers of x, we might obtain three equations for determining A, B, and C. But, as our equation must be true for all values of x, we can adopt a simpler process.

Make $x=1$; then $-3=2A$, and therefore $A=-\frac{3}{2}$.

Make $x=2$; then $-1=-B$, and therefore $B=1$.

Make $x=3$; then $1=2C$, and therefore $C=\frac{1}{2}$.

Putting these values in the assumed equation, and multiplying both its members by dx, we shall have

$$\frac{(2x-5)\,dx}{(x-1)(x-2)(x-3)} = -\frac{3}{2}\frac{dx}{x-1} + \frac{dx}{x-2} + \frac{1}{2}\frac{dx}{x-3},$$

and integrating,

$$\int \frac{(2x-5)\,dx}{(x-1)(x-2)(x-3)} =$$
$$-\tfrac{3}{2}\log(x-1)+\log(x-2)+\tfrac{1}{2}\log(x-3),$$

or finally

$$y=\log\frac{(x-2)\sqrt{x-3}}{\sqrt{(x-1)^3}}.$$

EXERCISES.

1. $dy=\dfrac{dx}{a^2-x^2}$, $\qquad y=\dfrac{1}{2a}\log\dfrac{a+x}{a-x}$.

2. $dy=\dfrac{(5x+1)\,dx}{x^2+x-2}$, $\qquad y=\log\{(x-1)^2(x+2)^3\}$.

3. $dy=\dfrac{(3x^2-1)\,dx}{x(x-1)(x+1)}$, $\quad y=\log(x^3-x)$.

68. *Second case: Some roots real and equal.* The existence of equal roots in the denominator entails the existence of some factors of the form $(x-a)^n$, n being the number of the equal roots, or of the equal factors. Now, in the decomposition of the given fraction, we must obtain as many partial fractions as there are factors in the denominator. If we were to assume, as we have done in the first case, fractions of the form $\dfrac{A}{x-a}$, $\dfrac{B}{x-a}$, $\dfrac{C}{x-a}$, . . . it is evident that these fractions would coalesce into a single fraction $\dfrac{A+B+C+\ldots}{x-a}$; and therefore, to obviate this, we shall assume the fractions

$$\frac{A}{(x-a)^n},\ \frac{B}{(x-a)^{n-1}},\ \frac{C}{(x-a)^{n-2}},\ \ldots\ \frac{F}{x-a}.$$

Let the given differential be

$$dy = \frac{(x^2+x)\,dx}{(x-2)^2\,(x-1)}.$$

Assume

$$\frac{x^2+x}{(x-2)^2\,(x-1)} = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{C}{x-1}.$$

By clearing of fractions, we find

$$x^2 + x = A\,(x-1) + B\,(x-1)\,(x-2) + C\,(x-2)^2.$$

This equation being true for all values of x, we make $x = 1$, and we find $C = 2$; then we make $x = 2$, and find $A = 6$; lastly we make $x = 0$, and replacing A and C by their values, we find $B = -1$.

Putting these values in our equation, and multiplying by dx, we have

$$\frac{(x^2+x)\,dx}{(x-2)^2\,(x-1)} = \frac{6dx}{(x-2)^2} - \frac{dx}{x-2} + \frac{2dx}{x-1}.$$

Hence

$$\int \frac{(x^2+x)\,dx}{(x-2)^2\,(x-1)} =$$

$$-\frac{6}{x-2} - \log\,(x-2) + 2\log\,(x-1),$$

and

$$y = \log\frac{(x-1)^2}{x-2} - \frac{6}{x-2}.$$

EXERCISES.

1. $dy = \dfrac{(x^2-2)\,dx}{x^2\,(x-1)}$, $\quad y = -\dfrac{2x+1}{x^2} + \log\dfrac{x}{x-1}.$

2. $dy = \frac{xdx}{(x+2)(x+3)^2}$, $y = -\frac{3}{x+3} + 2 \log \frac{x+3}{x+2}$.

3. $dy = \frac{(x+2)\,dx}{(x-1)^2(x-2)}$, $y = \frac{3}{x-1} + 4 \log \frac{x-2}{x-1}$.

69. *Third case: Unequal imaginary roots.* We know from Algebra that imaginary roots are found only in pairs, and that for each pair we must have a factor of the second degree which, when placed equal to zero, will give the imaginary roots. Thus $(x-a)^2 + b^2 = 0$ gives the two imaginary roots,

$$x = a + b\sqrt{-1}, \text{ and } x = a - b\sqrt{-1}.$$

In this third case, instead of assuming a partial fraction for each imaginary factor of the first degree, it is more convenient to assume for each pair of such factors (that is, for each factor of the second degree) a fraction of the form

$$\frac{A + Bx}{(x-a)^2 + b^2},$$

and to determine A and B by the ordinary process. Let, for instance,

$$dy = \frac{xdx}{(x+1)(x^2+1)}$$

Assume

$$\frac{x}{(x+1)(x^2+1)} = \frac{A + Bx}{x^2+1} + \frac{C}{x+1}.$$

Clearing of fractions, we find

$$x = Bx^2 + (A+B)\,x + A + Cx^2 + C.$$

Equating the coefficients of the like powers of x, we find

$$B + C = 0, \quad A + B = 1, \quad A + C = 0,$$

whence

$$A = \tfrac{1}{2}, \quad B = \tfrac{1}{2}, \quad C = -\tfrac{1}{2}.$$

Substituting these values in the assumed equation, and multiplying by dx, we have

$$\frac{xdx}{(x+1)(x^2+1)} = \frac{1}{2}\left(\frac{dx}{x^2+1} + \frac{xdx}{x^2+1} - \frac{dx}{x+1}\right),$$

and

$$\int \frac{xdx}{(x+1)(x^2+1)} = \tfrac{1}{2}\tan^{-1} x + \tfrac{1}{4}\log(x^2+1) - \tfrac{1}{2}\log(x+1),$$

or

$$y = \tfrac{1}{2}\left(\tan^{-1} x + \log \frac{\sqrt{x^2+1}}{x+1}\right).$$

EXERCISES.

1. $dy = \dfrac{x^2 dx}{(x^2+4)(x^2+1)}, \quad y = \tfrac{1}{3}(2\tan^{-1}\tfrac{1}{2}x - \tan^{-1} x).$

2. $dy = \dfrac{(x-2)\,dx}{(x-1)(x^2+1)},$

$$y = \tfrac{1}{2}\left(3\tan^{-1} x + \log \frac{\sqrt{x^2+1}}{x-1}\right).$$

3. $dy = \dfrac{5(x^2+1)\,dx}{(x+1)^2(x^2+4)},$

$$y = \log\{(x^2+4)(x+1)^2\} - \tfrac{2}{3}\tan^{-1}\frac{x}{2}.$$

70. ***Fourth case: Equal imaginary roots.*** In this case, if n be the number of the equal roots, we assume n fractions of the form

$$\frac{A+Bx}{[(x-a)^2+b^2]^n},\quad \frac{C+Dx}{[(x-a)^2+b^2]^{n-1}},\ \dots\ \frac{M+Nx}{(x-a)^2+b^2},$$

thus combining the methods of the second and third case. Let us have

$$dy=\frac{x^3dx}{(x^2+1)^2}.$$

Assume

$$\frac{x^3}{(x^2+1)^2}=\frac{A+Bx}{(x^2+1)^2}+\frac{C+Dx}{x^2+1}.$$

Clearing of fractions, we obtain

$$x^3=A+Bx+Cx^2+Dx^3+C+Dx,$$

which gives

$$A=0,\quad B=-1,\quad C=0,\quad D=1.$$

Substituting these values in the assumed equation and multiplying by dx, we have

$$\frac{x^3dx}{(x^2+1)^2}=-\frac{xdx}{(x^2+1)^2}+\frac{xdx}{x^2+1};$$

whence

$$\int\frac{x^3dx}{(x^2+1)^2}=\frac{1}{2}\cdot\frac{1}{x^2+1}+\tfrac{1}{2}\log(x^2+1),$$

or

$$y=\frac{1}{2(x^2+1)}+\log\sqrt{x^2+1}.$$

EXERCISE.

$$dy = \frac{(x-2)\,dx}{(x^2+2)^2\,(x-1)},$$

$$y = \frac{2x-3}{2(x^2+2)} + \sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}} + \log\frac{\sqrt{x^2+2}}{x-1}.$$

We conclude that all differentials which are rational fractions can be integrated, provided the factors of the denominator can be discovered. Their integrals will depend on one or more of the forms

$$\int\frac{dx}{x+a},\qquad \int x^m dx,$$

$$\int\frac{x\,dx}{(x^2+a^2)^n},\qquad \int\frac{dx}{(x^2+a^2)^n}.$$

Integration of Binomial Differentials.

71. Every binomial differential can be reduced to the form

$$dy = A\,(a+bx^n)^{\frac{r}{s}}\,x^{m-1}\,dx,$$

in which m, n, r, and s are whole numbers, and n positive. The reduction to this form is made by different means, according to each particular case, especially by the use of an auxiliary variable. The factor A may be omitted during the integration, as it represents a constant.

There are three cases in which the integration of a binomial of the above form can be easily performed.

The first case is when $\frac{r}{s}$ is a whole number.

Then the binomial is rational, and can be integrated by some of the processes already explained.

The second case is when $\frac{m}{n}$ is a whole number. For, in this case, if we assume

$$a + bx^n = z^s, \text{ or } z = (a + bx^n)^{\frac{1}{s}},$$

we shall have

$$x = \left(\frac{z^s - a}{b}\right)^{\frac{1}{n}}, \text{ and } x^m = \left(\frac{z^s - a}{b}\right)^{\frac{m}{n}},$$

whence, differentiating, and dividing by m,

$$x^{m-1}\,dx = \frac{s}{bn}\left(\frac{z^s - a}{b}\right)^{\frac{m}{n}-1} z^{s-1}\,dz;$$

and therefore

$$(a + bx^n)^{\frac{r}{s}}\,x^{m-1}\,dx = \frac{s}{bn}\left(\frac{z^s - a}{b}\right)^{\frac{m}{n}-1} z^{r+s-1}\,dz, \quad (31)$$

which expression is rational and integrable, if $\frac{m}{n}$ is a whole number.

The third case is when the sum $\frac{m}{n} + \frac{r}{s}$ is a whole number. For, in this case, if we assume

$$a + bx^n = z^s x^n, \text{ or } z = \left(\frac{a + bx^n}{x^n}\right)^{\frac{1}{s}},$$

we shall have

$$x = \left(\frac{a}{z^s - b}\right)^{\frac{1}{n}} = a^{\frac{1}{n}}(z^s - b)^{-\frac{1}{n}},$$

and $$x^m = a^{\frac{m}{n}} (z^s - b)^{-\frac{m}{n}},$$

whence, differentiating, and dividing by m,

$$x^{m-1}\,dx = -\frac{s}{n} a^{\frac{m}{n}} (z^s - b)^{-\frac{m}{n}-1} z^{s-1}\,dz;$$

and therefore

$$(a + bx^n)^{\frac{r}{s}} x^{m-1}\,dx = -\frac{s}{n} a^{\frac{m}{n}+\frac{r}{s}} (z^s - b)^{-\frac{m}{n}-\frac{r}{s}-1} z^{r+s-1}\,dz, \quad (32)$$

which expression is rational and integrable when $\frac{m}{n} + \frac{r}{s}$ is a whole number.

Integration by Successive Reduction.

72. When the proposed binomial does not fall under one of the three cases just mentioned, and when its integration can be made to depend on that of a simpler expression, we can, by a successive repetition of the process, make it finally depend on an expression integrable by some fundamental formula. This is the process by which irrational differentials are integrated when they do not allow of immediate and direct integration. It is called the process of *successive reduction.*

Let $$(a + bx^n)^p x^{m-1}\,dx$$

be an irrational binomial differential, p being a fractional exponent, and m and n any whole numbers. The difficulty of proceeding to its integration arises from the value of the exponents, which may be either too great or too small. In the former case we must cause the integration to depend on lower exponents; in the latter we must try to make it depend on higher ones. Let us come to particulars.

First case: To lower the exponent m − 1 *of the variable.*

We have identically

$$(a+bx^n)^p\, x^{m-1}\, dx = x^{m-n}\,(a+bx^n)^p\, x^{n-1}\, dx.$$

Making

$$x^{m-n} = u, \text{ and } (a+bx^n)^p\, x^{n-1}\, dx = dv,$$

we shall have

$$du = (m-n)\, x^{m-n-1}\, dx,\; v = \frac{(a+bx^n)^{p+1}}{nb\,(p+1)},$$

and, integrating by formula (30),

$$\int (a+bx^n)^p\, x^{m-1}\, dx =$$
$$\frac{x^{m-n}\,(a+bx^n)^{p+1}}{nb\,(p+1)} - \int \frac{(a+bx^n)^{p+1}}{nb\,(p+1)}\,(m-n)\, x^{m-n-1}\, dx,$$

or, making, for greater simplicity, $a+bx^n = X$, and consequently

$$X^{p+1} = aX^p + bx^nX^p,$$

$$\int X^p\, x^{m-1}\, dx = \frac{X^{p+1}\, x^{m-n}}{nb\,(p+1)}$$
$$-\frac{a\,(m-n)}{nb(p+1)}\int X^p\, x^{m-n-1}\, dx - \frac{b\,(m-n)}{nb(p+1)}\int X^p\, x^{m-1} dx.$$

Transposing the last term to the first member, and reducing, we get

$$\frac{b\,(np+m)}{nb\,(p+1)}\int X^p\, x^{m-1}\, dx =$$
$$\frac{X^{p+1}\, x^{m-n}}{nb\,(p+1)} - \frac{a\,(m-n)}{nb\,(p+1)}\int X^p\, x^{m-n-1}\, dx,$$

and dividing by the coefficient of the first member,

$$\int X^p x^{m-1} dx =$$
$$\frac{X^{p+1} x^{m-n}}{b(np+m)} - \frac{a(m-n)}{b(np+m)} \int X^p x^{m-n-1} dx. \quad (33)$$

By this formula the integral of $X^p x^{m-1} dx$ is made to depend on that of $X^p x^{m-n-1} dx$, where the exponent of the variable is diminished by n units at each application. The formula fails when $np + m = 0$; but then the integration can be made by formula (32).

Second case: To lower the exponent p *of the binomial.*

We have identically

$$(a + bx^n)^p = a(a + bx^n)^{p-1} + bx^n (a + bx^n)^{p-1},$$

or

$$X^p = aX^{p-1} + bx^n X^{p-1},$$

whence

$$\int X^p x^{m-1} dx = a \int X^{p-1} x^{m-1} dx + b \int X^{p-1} x^{m+n-1} dx.$$

Now, the last term of this equation can be reduced by formula (33); for, changing m into $m + n$, and p into $p - 1$, and multiplying by b, that formula gives

$$b \int X^{p-1} x^{m+n-1} dx =$$
$$\frac{X^p x^m}{np+m} - \frac{am}{np+m} \int X^{p-1} x^{m-1} dx.$$

Substituting this for the last term of the above equation, uniting the similar terms, and reducing, we obtain

$$\int X^p x^{m-1}\,dx = \frac{X^p x^m}{np+m} + \frac{anp}{np+m}\int X^{p-1} x^{m-1}\,dx. \quad (34)$$

By this formula the integral of $X^p x^{m-1}\,dx$ is made to depend on that of $X^{p-1} x^{m-1}\,dx$, where the exponent of the binomial is diminished by unity at each application. When $np + m = 0$, this formula fails; but then the integration can be made by formula (32), as already remarked.

Third case: To increase the exponent of the variable.

When m is negative we may need to diminish it arithmetically or increase it algebraically. To do this we proceed as follows. Reversing formula (33), we obtain

$$\int X^p x^{m-n-1}\,dx = \frac{X^{p+1} x^{m-n}}{a(m-n)} - \frac{b(np+m)}{a(m-n)}\int X^p x^{m-1}\,dx.$$

This equation, by changing m into $-m+n$, will become

$$\int X^p x^{-m-1}\,dx = -\frac{X^{p+1} x^{-m}}{am} + \frac{b(np-m+n)}{am}\int X^p x^{-m+n-1}\,dx. \quad (35)$$

By this formula the integral of $X^p x^{-m-1}\,dx$ is made to depend on that of $X^p x^{-m+n-1}\,dx$, where the exponent of the variable is increased by n units at each application.

Fourth case: To increase the exponent of the binomial.

When p is negative, and we need to increase it, we proceed as follows. Reversing formula (34), we obtain

$$\int X^{p-1} x^{m-1} dx = -\frac{X^p x^m}{anp} + \frac{np+m}{anp}\int X^p x^{m-1} dx.$$

This equation, by changing p into $-p+1$, will become

$$\int X^{-p} x^{m-1} dx =$$
$$\frac{X^{-p+1} x^m}{an(p-1)} - \frac{m+n-np}{an(p-1)}\int X^{-p+1} x^{m-1} dx. \quad (36)$$

By this formula the integral of $X^{-p} x^{m-1} dx$ is made to depend on that of $X^{-p+1} x^{m-1} dx$, where the exponent of the binomial is increased by unity at each application.

73. The integration of the expression

$$\frac{x^q dx}{\sqrt{2rx - x^2}}$$

can be made to depend on that of a like expression, in which the exponent q is diminished by unity. We have identically

$$\frac{x^q dx}{\sqrt{2rx - x^2}} = x^q\, dx\, (2rx - x^2)^{-\frac{1}{2}} = (2r-x)^{-\frac{1}{2}} x^{q-\frac{1}{2}} dx.$$

Now, representing $(2r-x)^{-\frac{1}{2}}$ by $X^{-\frac{1}{2}}$, and comparing with formula (33), we have

$$a = 2r,\ b = -1,\ n = 1,\ p = -\tfrac{1}{2},\ m = q + \tfrac{1}{2};$$

whence

$$b\,(np+m) = -\,q,\ \ a\,(m-n) = r\,(2q-1),$$
$$p+1 = \tfrac{1}{2},\ \ m-n = q-\tfrac{1}{2}.$$

Substituting these values in (33), we find

$$\int X^{-\frac{1}{2}}\, x^{q-\frac{1}{2}}\, dx =$$
$$-\frac{X^{\frac{1}{2}}\, x^{q-\frac{1}{2}}}{q} + \frac{r\,(2q-1)}{q}\int X^{-\frac{1}{2}}\, x^{q-1-\frac{1}{2}}\, dx,$$

that is,

$$\int \frac{x^q\, dx}{\sqrt{2rx-x^2}} =$$
$$-\frac{x^{q-1}\sqrt{2rx-x^2}}{q} + \frac{r\,(2q-1)}{q}\int\frac{x^{q-1}\, dx}{\sqrt{2rx-x^2}}\,. \qquad (37)$$

If q be entire and positive, by a successive application of this formula, we ultimately arrive at the form

$$\int\frac{dx}{\sqrt{2rx-x^2}} = \text{vers}^{\,1}\frac{x}{r}\,.$$

In like manner the integration of the expression

$$\frac{x^q\, dx}{\sqrt{2rx+x^2}}$$

can be made to depend on that of a like expression in which the exponent q is diminished by unity. For the only difference between this case and the preceding one is that b is now positive instead of negative, and therefore $b\ (np+m)=q$; which shows that the signs of the second member of the formula ought to be changed. Accordingly

$$\int \frac{x^q\,dx}{\sqrt{2rx+x^2}} =$$

$$\frac{x^{q-1}\sqrt{2rx+x^2}}{q} - \frac{r(2q-1)}{q}\int \frac{x^{q-1}\,dx}{\sqrt{2rx+x^2}}. \quad (38)$$

Formulas (33), (34), (35), (36), (37), (38) are called *formulas of reduction*, and have a wide range of application.

EXAMPLES.

1. To integrate

$$dy = \frac{x^3 dx}{\sqrt{r^2-x^2}}.$$

Here we have

$X = r^2 - x^2$, $a = r^2$, $b = -1$, $n = 2$, $p = -\frac{1}{2}$, $m = 3$.

By formula (33) we have, for this case,

$$\int X^{-\frac{1}{2}} x^2\,dx = \frac{X^{\frac{1}{2}}x}{-2} - \frac{r^2}{-2}\int X^{-\frac{1}{2}}\,dx,$$

that is,

$$\int \frac{x^2\,dx}{\sqrt{r^2-x^2}} = -\frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\int \frac{dx}{\sqrt{r^2-x^2}}$$

or

$$\int \frac{x^2\,dx}{\sqrt{r^2-x^2}} = -\frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}.$$

2. To integrate

$$dy = x^2\,dx\,\sqrt{r^2-x^2}.$$

Here we have

$X = r^2 - x^2$, $a = r^2$, $b = -1$, $n = 2$, $p = \frac{1}{2}$, $m = 3$.

Formula (33) gives, in this case,

$$\int X^{\frac{1}{2}} x^2\, dx = \frac{X^{\frac{3}{2}} x}{-4} - \frac{r^2}{-4}\int X^{\frac{1}{2}}\, dx,$$

that is,

$$\int x^2\, dx\, \sqrt{r^2 - x^2} = -\frac{x\sqrt{(r^2 - x^2)^3}}{4} + \frac{r^2}{4}\int dx\, \sqrt{r^2 - x^2}.$$

To integrate the last term of this equation we use formula (34), in which we shall make

$$X = r^2 - x^2,\ a = r^2,\ n = 2,\ p = \tfrac{1}{2},\ m = 1.$$

We obtain

$$\int X^{\frac{1}{2}}\, dx = \frac{X^{\frac{1}{2}} x}{2} + \frac{r^2}{2}\int X^{-\frac{1}{2}}\, dx,$$

that is,

$$\int dx\, \sqrt{r^2 - x^2} = \frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}.$$

Substituting this in the preceding equation, we have

$$\int x^2\, dx\, \sqrt{r^2 - x^2} =$$

$$-\frac{x}{4}\sqrt{(r^2 - x^2)^3} + \frac{r^2 x}{8}\sqrt{r^2 - x^2} + \frac{r^4}{8}\sin^{-1}\frac{x}{r}.$$

3. To integrate

$$dy = \frac{dx}{(1 + x^2)^2}.$$

Here we have

$$X = 1 + x^2,\ a = 1,\ n = 2,\ -p = -2,\ m = 1.$$

By formula (36) we have

$$\int X^{-2}\,dx = \frac{X^{-1}x}{2} + \frac{1}{2}\int X^{-1}\,dx,$$

that is,

$$\int dx\,(1+x^2)^{-2} = \frac{x}{2}\,(1+x^2)^{-1} + \frac{1}{2}\int dx\,(1+x^2)^{-1},$$

or

$$\int \frac{dx}{(1+x^2)^2} = \frac{x}{2\,(1+x^2)} + \frac{1}{2}\tan^{-1}x.$$

Integration of some Trinomial Differentials.

74. A trinomial differential of the form

$$(a + bx \pm x^2)^{\frac{p}{2}}\,x^m\,dx$$

can be made rational in terms of an auxiliary variable, when p and m are whole numbers. When p is even the expression is already rational; when p is odd, then, making $p = 2n + 1$, the expression becomes

$$(a + bx \pm x^2)^n\,(a + bx \pm x^2)^{\frac{1}{2}}\,x^m\,dx,$$

in which the only irrational part is $\sqrt{a + bx \pm x^2}$.

When x^2 is positive we obtain a rational form by assuming

$$\sqrt{a + bx + x^2} = z - x,$$

from which we shall get

$$a + bx = z^2 - 2zx,$$

$$x = \frac{z^2 - a}{2z + b}, \quad dx = \frac{2\,(z^2 + bz + a)}{(2z + b)^2}\,dz,$$

and $$\sqrt{a+bx+x^2} = \frac{z^2+bz+a}{2z+b}.$$

Thus the given expression will become rational in terms of z, and therefore integrable. After the integration it only remains to substitute for z its value

$$x + \sqrt{a+bx+x^2}.$$

When x^2 is negative let us assume

$$\sqrt{a+bx-x^2} = \sqrt{-(x-h)(x-k)} = (x-h)\,z,$$

where h and k are the roots of the equation

$$x^2 - bx - a = 0.$$

We then have

$$a + bx - x^2 = (x-h)(k-x) = (x-h)^2 z^2,$$

or

$$k - x = (x-h)\,z^2.$$

Hence

$$z = \sqrt{\frac{k-x}{x-h}}, \quad x = \frac{k+hz^2}{1+z^2},$$

$$dx = -\frac{2(k-h)\,zdz}{(1+z^2)^2},$$

and

$$\sqrt{a+bx-x^2} = \left(\frac{k+hz^2}{1+z^2} - h\right) z = \frac{(k-h)\,z}{1+z^2}.$$

Thus the given expression will become rational and integrable in terms of z. After the integration it will only remain to substitute for z its value

$$\sqrt{\frac{k-x}{x-h}}.$$

EXERCISES.

1. $dy = \frac{dx}{\sqrt{1+x+x^2}}, \quad y = \log(2x+1+2\sqrt{1+x+x^2}).$

2. $dy = \frac{dx}{\sqrt{2-x-x^2}}, \quad y = -2\tan^{-1}\sqrt{\frac{1-x}{x+2}}.$

Integration by Series.

75. When an expression Xdx, in which X is a function of x, is to be integrated, it is often convenient and useful to develop X into a series by any of the known methods, and then to integrate each term separately. This is called *integration by series*. If the series obtained be convergent for any particular value of x, we shall obtain the approximate value of the integral for that particular value of x.

Thus, given

$$dy = \frac{dx}{1+x},$$

we may, either by the binomial formula or by mere division, obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots$$

Whence

$$\frac{dx}{1+x} = dx - xdx + x^2dx - x^3dx + x^4dx - \ldots$$

and

$$y = \int \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

which is the expression for $\log(1+x)$.

Again, given

$$dy = \frac{dx}{1+x^2},$$

we may, by simple division, obtain the series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots,$$

whence

$$\frac{dx}{1+x^2} = dx - x^2dx + x^4dx - x^6dx + x^8dx - \ldots,$$

and

$$y = \int \frac{dx}{1+x^2} = 1 - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots,$$

which is the expression for $\tan^{-1} x$.

In a similar manner, if we have

$$dy = \frac{dx}{\sqrt{1-x^2}},$$

we may obtain, by the binomial formula,

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} =$$

$$1 + \frac{x^2}{2} + \frac{3x^4}{2.4} + \frac{3.5x^6}{2.4.6} + \frac{3.5.7x^8}{2.4.6.8} + \ldots,$$

whence, multiplying by dx, and integrating, we find

$$y = \int \frac{dx}{\sqrt{1-x^2}} =$$

$$x + \frac{x^3}{2.3} + \frac{3x^5}{2.4.5} + \frac{3.5x^7}{2.4.6.7} + \frac{3.5.7x^9}{2.4.6.8.9} + \ldots$$

which is the expression for $\sin^{-1} x$.

76. Let us have to integrate the general expression

$$dy = Xdx,$$

where X is a function of x. Integrating by parts, we have

$$y = Xx - \int xdX = Xx - \int \frac{dX}{dx} \cdot xdx.$$

This last term, again integrated by parts, will give

$$\int \frac{dX}{dx} \cdot xdx = \frac{x^2}{2} \frac{dX}{dx} - \int \frac{x^2}{2} \cdot \frac{d^2X}{dx}.$$

This last term will give, by the same method of integration,

$$\int \frac{x^2}{2} \cdot \frac{d^2X}{dx} = \int \frac{x^2}{2} dx \frac{d^2X}{dx^2} = \frac{x^3}{2.3} \frac{d^2X}{dx^2} - \int \frac{x^3}{2.3} \cdot \frac{d^3X}{dx^3},$$

and continuing to integrate in the same manner, we shall find

$$y = \int Xdx = Xx - \frac{dX}{dx} \cdot \frac{x^2}{2} + \frac{d^2X}{dx^2} \cdot \frac{x^3}{2.3} - \frac{d^3X}{dx^3} \cdot \frac{x^4}{2.3.4} + \dots$$

This elegant formula, by which the integral is obtained through successive differentiations, was discovered by John Bernouilli, and bears his name.

Integration of Trigonometric Expressions.

77. Trigonometric expressions can be reduced to integrable forms by suitable transformations based on the correlation of trigonometric lines.

1. Given $$dy = \frac{dx}{\sin x},$$

we have from trigonometry

$$\sin x = 2 \sin \tfrac{1}{2}x \cos \tfrac{1}{2}x = 2 \tan \tfrac{1}{2}x \cos^2 \tfrac{1}{2}x;$$

whence

$$\frac{dx}{\sin x} = \frac{dx}{2 \tan \frac{1}{2}x \cos^2 \frac{1}{2}x} = \frac{1}{\tan \frac{1}{2}x} \times \frac{\frac{1}{2}dx}{\cos^2 \frac{1}{2}x} = \frac{d(\tan \frac{1}{2}x)}{\tan \frac{1}{2}x};$$

and therefore

$$y = \int \frac{dx}{\sin x} = \log (\tan \tfrac{1}{2}x). \qquad (39)$$

2. Given $$dy = \frac{dx}{\cos x},$$

we have $\cos x = \sin (90° - x)$. Hence, substituting $90° - x$ for x in (39), we shall have

$$y = \int \frac{dx}{\cos x} = -\log [\tan (45° - \tfrac{1}{2}x)]. \qquad (40)$$

3. Given $$y = \frac{dx}{\tan x},$$

we have

$$\frac{dx}{\tan x} = \frac{\cos x dx}{\sin x} = \frac{d(\sin x)}{\sin x};$$

and therefore

$$y = \int \frac{dx}{\tan x} = \log (\sin x). \qquad (41)$$

4. Given $$y = \frac{dx}{\cot x},$$

we have

$$\frac{dx}{\cot x} = \frac{\sin x dx}{\cos x} = -\frac{d(\cos x)}{\cos x};$$

whence

$$y = \int \frac{dx}{\cot x} = -\log(\cos x). \qquad (42)$$

5. Given

$$dy = \frac{dx}{\sin x \cos x},$$

we have

$$\sin x \cos x = \tfrac{1}{2} \sin 2x.$$

Substituting, and integrating,

$$y = \int \frac{dx}{\sin x \cos x} = \int \frac{2dx}{\sin 2x} = \log(\tan x).$$

6. Given

$$dy = \sin^m x \cos^n x dx,$$

it will be convenient to transform this differential into an algebraic expression by assuming $\sin x = z$, whence $\cos x = \sqrt{1 - z^2}$, and

$$dx = \frac{dz}{\sqrt{1 - z^2}}.$$

Thus we shall obtain the differential in the form

$$dy = (1 - z^2)^{\frac{n-1}{2}} z^m dz,$$

which can always be integrated when $m + n$ is a whole and *even* number (No. 71).

If $m + n$ is a whole but *uneven* number, then the integration may be made by formula (33) or

(34), by which the exponents will be gradually reduced till we reach some simple and elementary form.

Formula (33) will give us

$$\int (1-z^2)^{\frac{n-1}{2}} z^m\, dz =$$

$$-\frac{z^{m-1}(1-z^2)^{\frac{n+1}{2}}}{m+n} + \frac{m-1}{m+n}\int (1-z^2)^{\frac{n-1}{2}} z^{m-2}\, dz,$$

or

$$\int \sin^m x \cos^n x dx =$$

$$-\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x dx;$$

from which, if $n=0$, we obtain

$$\int \sin^m x dx =$$

$$-\frac{\sin^{m-1} x \cos x}{m} + \frac{m-1}{m}\int \sin^{m-2} x dx.$$

Formula (34) will give us

$$\int (1-z^2)^{\frac{n-1}{2}} z^m\, dz =$$

$$\frac{z^{m+1}(1-z^2)^{\frac{n-1}{2}}}{m+n} + \frac{n-1}{m+n}\int (1-z^2)^{\frac{n-3}{2}} z^m\, dz,$$

or

$$\int \sin^m x \cos^n x dx =$$

$$\frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x dx;$$

from which, if $m = 0$, we obtain

$$\int \cos^n x dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x dx.$$

Integration of Logarithmic Differentials.

78. Let $\qquad dy = x^{m-1} (\log x)^n dx$

be proposed for integration. Assuming $(\log x)^n = u$, and $x^{m-1} dx = dv$, we obtain

$$du = \frac{n (\log x)^{n-1}}{x} dx, \text{ and } v = \frac{x^m}{m}.$$

Hence, by formula (30),

$$\int x^{m-1} (\log x)^n dx = \frac{x^m}{m} (\log x)^n - \frac{n}{m}\int x^{m-1} (\log x)^{n-1} dx. \qquad (43)$$

By this formula we can reduce the exponent of $\log x$ by 1 at each application. The formula fails for $m = 0$; but then the integration is obvious.

If we reverse (43), we obtain

$$\int x^{m-1} (\log x)^{n-1} dx = \frac{x^m}{n} (\log x)^n - \frac{m}{n}\int x^{m-1} (\log x)^n dx,$$

and, if we change $n-1$ into $-n$, we shall have

$$\int \frac{x^{m-1} dx}{(\log x)^n} = -\frac{x^m}{(n-1)(\log x)^{n-1}} + \frac{m}{n-1}\int \frac{x^{m-1} dx}{(\log x)^{n-1}}, \qquad (44)$$

and by this formula we are enabled to reduce the exponent of the denominator by 1 at each application. The formula fails when $n = 1$; but when $m = 0$ and $n = 1$, the integral becomes

$$\int \frac{dx}{x \log x} = \frac{d.\log x}{\log x} = \log (\log x).$$

To integrate $dy = \frac{dx}{\log x}$; it suffices to make $\log x = z$, whence $x = e^z$, $dx = e^z\, dz$; and $dy = \frac{e^z\, dz}{z}$. And, as $e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{2.3} + \ldots$, the integration is obvious.

Integration of Exponential Differentials.

79. Let $dy = x^m a^x\, dx$

be proposed for integration. Assuming $x^m = u$, $a^x dx = dv$, we obtain $du = mx^{m-1}\, dx$, $v = \frac{a^x}{\log a}$; and formula (30) will give

$$\int x^m a^x\, dx = \frac{x^m a^x}{\log a} - \frac{m}{\log a} \int x^{m-1} a^x\, dx. \qquad (45)$$

At each application of this formula the exponent of x will be lowered by 1, till we reach the simple form

$$\int a^x\, dx.$$

When m is negative, by reversing the formula we first obtain

$$\int x^{m-1} a^x\, dx = \frac{x^m a^x}{m} - \frac{\log a}{m} \int x^m a^x\, dx,$$

then, replacing $m - 1$ by $-m$, and reducing,

$$\int \frac{a^x\, dx}{x^m} = -\frac{a^x}{(m-1)\, x^{m-1}} + \frac{\log a}{m-1} \int \frac{a^x\, dx}{x^{m-1}}. \qquad (46)$$

By this formula the exponent of the denominator is lowered by 1 at each application. But the formula fails when $m=1$; and in this case the integral of $\frac{a^x\,dx}{x}$ is obtained by changing a^x into its development

$$1+(\log a)\,x+(\log a)^2\frac{x^2}{2}+(\log a)^3\frac{x^3}{2.3}+\,.\,.\,.$$

80. Let $\qquad dy=Xa^x\,dx,$

X being a function of x. Integrating by parts, we have

$$\int Xa^x\,dx=\frac{Xa^x}{\log a}-\int\frac{a^x}{\log a}\,dX.$$

If we take the successive differentials of X, and place

$$dX=X'\,dx,\ dX'=X''\,dx,\ dX''=X'''\,dx,\ .\,.\,.$$

we obtain

$$\int\frac{a^x}{\log a}\,dX=\frac{X'\,a^x}{(\log a)^2}-\int\frac{a^x}{(\log a)^2}\,dX',$$

$$\int\frac{a^x}{(\log a)^2}\,dX'=\frac{X''\,a^x}{(\log a)^3}-\int\frac{a^x}{(\log a)^3}\,dX'',$$

and so on. Hence, by successive substitutions, we shall find

$$\int Xa^x\,dx=$$

$$a^x\left\{\frac{X}{\log a}-\frac{X'}{(\log a)^2}+\frac{X''}{(\log a)^3}-\,.\,.\,.\,\pm\frac{X^{(n)}}{(\log a)^{n+1}}\right\}$$

$$\mp\int\frac{a^x\,dX^{(n)}}{(\log a)^{n+1}}. \qquad (47)$$

If X is such a function that one of its differential coefficients X', X'', . . . is constant, then the next differential coefficient will be $=0$, and the corresponding term $\mp\int\frac{a^x\,dX^{(n)}}{(\log a)^{n+1}}$ will vanish. In such a case the integral will be exact. Thus, if we have

$$dy=(x^2-h^2)\,e^x\,dx,$$

then $X=x^2-h^2$, $dX=2xdx$, $X'=2x$, $dX'=2dx$, $X''=2$, $dX''=0$. On the other hand, $\log e=1$. Substituting in (47), we find

$$y=e^x\,(x^2-h^2-2x+2).$$

Integration of Total Differentials of the First Order.

81. The total differential of a function $u=f(x,y)$ is, as we have seen (No. 19),

$$du=\frac{du}{dx}\,dx+\frac{du}{dy}\,dy,$$

in which $\frac{du}{dx}\,dx$ and $\frac{du}{dy}\,dy$ are the partial differentials of the function. We have also seen (No. 25, *Scholium*) that the existence of an exact total differential of such a function entails the existence of the relation

$$\frac{d\left(\frac{du}{dx}\right)}{dy}=\frac{d\left(\frac{du}{dy}\right)}{dx};$$

whence it follows that a differential of the form

$$du=Mdx+Ndy$$

will be an exact total differential when M and N are such as will satisfy the condition $\frac{dM}{dy}=\frac{dN}{dx}$; for, in such a case, we evidently have $M=\frac{du}{dx}$, and $N=\frac{du}{dy}$.

If the condition is satisfied we may immediately integrate the partial differential Mdx and write

$$u=\int Mdx+Y,$$

Y exhibiting a function of y which is to be determined so as to satisfy the condition $\frac{du}{dy}=N$. Accordingly, we now differentiate our integral with respect to y, and, dividing by dy, we have

$$\frac{du}{dy}=\frac{d\int Mdx}{dy}+\frac{dY}{dy}=N,$$

whence

$$\frac{dY}{dy}=N-\frac{d\int Mdx}{dy},$$

and

$$Y=\int\left(N-\frac{d\int Mdx}{dy}\right)dy,$$

and finally

$$u=\int Mdx+\int\left(N-\frac{d\int Mdx}{dy}\right)dy. \qquad (48)$$

EXAMPLE.

Let

$$du = (2axy - 3bx^2y)\,dx + (ax^2 - bx^3)\,dy.$$

Here we have

$$M = 2axy - 3bx^2y, \quad N = ax^2 - bx^3,$$

whence

$$\frac{dM}{dy} = 2ax - 3bx^2 = \frac{dN}{dx}.$$

Hence our total differential is exact and immediately integrable. Integrating the partial differential Mdx, we have

$$u = \int (2axy - 3bx^2y)\,dx + Y = ax^2y - bx^3y + Y.$$

Differentiating this integral with respect to y, we find

$$\frac{du}{dy} = ax^2 - bx^3 + \frac{dY}{dy} = N,$$

whence

$$\frac{dY}{dy} = N - (ax^2 - bx^3) = 0, \text{ and } Y = C.$$

And therefore

$$u = ax^2y - bx^3y + C.$$

EXERCISES.

1. $du = 3x^2y^2dx + 2x^3ydy$, $\quad u = x^3y^2 + C.$

2. $du = \dfrac{dx}{y} + \left(2y - \dfrac{x}{y^2}\right) dy$, $\quad u = \dfrac{x}{y} + y^2 + C.$

3. $du = \dfrac{xdy - ydx}{y^2 + x^2}$, $\quad u = \tan^{-1}\dfrac{y}{x} + C.$

Integration of the Equation $Mdx + Ndy = 0$.

82. The implicit function $f(x, y) = 0$ has for its differential (No. 21)

$$\frac{df}{dx}dx + \frac{df}{dy}dy = 0,$$

where $\frac{df}{dx}$ and $\frac{df}{dy}$ are the differential coefficients of the function taken with respect to x and y. Representing $\frac{df}{dx}$ by M, and $\frac{df}{dy}$ by N, the differential will take the form

$$Mdx + Ndy = 0.$$

Now, this equation, whenever we have $\frac{dM}{dy} = \frac{dN}{dx}$, will be an exact differential (No. 81), and may be integrated by formula (48); but, as we have here $du = 0$, the integral will be $u = C$.

When by any transformation the equation can be placed under the form

$$Xdx + Ydy = 0,$$

X being a function of x alone, and Y a function of y alone, the integral can be found by taking the sum of the integrals of the two terms. Thus

$$\int Xdx + \int Ydy = C.$$

When the equation can be placed under the form

$$Ydx + Xdy = 0,$$

or under the form

$$XYdx + X'Y'dy = 0,$$

the variables can be separated by division. Thus

$$\frac{dx}{X}+\frac{dy}{Y}=0, \quad \text{and} \quad \frac{X}{X'}\,dx+\frac{Y'}{Y}\,dy=0,$$

and when the variables are thus separated the integration becomes possible. Hence the separation of the variables has been one of the main objects of study on the part of mathematicians.

EXAMPLES.

1. Given $ydx - xdy = 0$. Divide by xy. Then

$$\frac{dx}{x}-\frac{dy}{y}=0, \quad \log x - \log y = C = \log c;$$

and therefore

$$\frac{x}{y}=c, \text{ or } x=cy.$$

2. Given $xy^2dx + dy = 0$. Divide by y^2. Then

$$xdx+\frac{dy}{y^2}=0, \quad \frac{x^2}{2}-\frac{1}{y}=C,$$

whence

$$x^2y=2\,(Cy+1).$$

3. Given $(1-x^2)\;ydx - (1-y^2)\;x^2dy=0$. Divide by x^2y. Then

$$\frac{1-x^2}{x^2}\,dx-\frac{1-y^2}{y}\,dy=0,$$

or

$$\frac{dx}{x^2}-dx-\frac{dy}{y}+ydy=0,$$

whence

$$-\frac{1}{x} - x - \log y + \frac{y^2}{2} = C,$$

or

$$y^2 - \log(y^2) = \frac{2(1+x^2)}{x} + C'.$$

83. When the equation $Mdx + Ndy = 0$ is homogeneous with regard to the variables, that is, when the sum of the exponents of the variables is the same in M as in N, the variables can be separated by the aid of an auxiliary variable. Let

$$x^2dy - y(x+y)\,dx = 0.$$

This equation being homogeneous, we assume $y = zx$, and therefore $dy = zdx + xdz$. Substituting these values in the equation, we have

$$x^2zdx + x^3dz - (z^2x^2 + x^2z)\,dx = 0,$$

and, dividing by x^2,

$$xdz - z^2dx = 0, \text{ and } \frac{dz}{z^2} = \frac{dx}{x};$$

whence

$$\log x = -\frac{1}{z} + C = C - \frac{x}{y}; \text{ and } y = \frac{x}{C - \log x}.$$

In like manner, the equation

$$\frac{x^2 + xy}{x - y}\,dy - ydx = 0$$

being homogeneous, we assume

$$y = zx, \quad dy = zdx + xdz,$$

and we find, by substitution and reduction,

$$x(1+z)\,dz + 2z^2dx = 0, \text{ whence } \frac{dx}{x} = -\frac{1+z}{2z^2}\,dz,$$

and

$$\log x = \frac{1}{2z} - \tfrac{1}{2}\log z = \frac{x}{2y} - \log\sqrt{\frac{y}{x}} + C.$$

Again, the equation

$$xdy - ydx = dx\sqrt{x^2+y^2}$$

being homogeneous, we assume

$$y = zx, \quad dy = zdx + xdz,$$

and we find, by substitution and reduction,

$$xdz = dx\sqrt{1+z^2}, \text{ whence } \frac{dz}{\sqrt{1+z^2}} = \frac{dx}{x}.$$

Accordingly

$$\log x = \log\left(z + \sqrt{1+z^2}\right) = \log\left(\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right) + C,$$

an integral which, freed from transcendentals and radicals, reduces to

$$x^2 = 2C_1y + C_1^2,$$

C_1 being an arbitrary constant.

84. When $Mdx + Ndy = 0$ is not an exact differential, it is possible to reduce it to an exact differential by an *integrating factor*. Thus

$$(1+y^2)\,dx + xydy = 0$$

becomes an exact differential if it be multiplied by $2x$, and gives the integral

$$x^2(1+y^2) = C.$$

The multiplier $2x$ is termed an *integrating factor*. The same equation might have been made an exact differential by the integrating factor $\frac{1}{x(1+y^2)}$, whence

$$\frac{dx}{x}+\frac{ydy}{1+y^2}=0, \text{ and } \log x+\tfrac{1}{2}\log(1+y^2)=C;$$

a result identical with the preceding one, though under a different form.

The expression $Mdx+Ndy$ can be written as follows:

$$\tfrac{1}{2}(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+\tfrac{1}{2}(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right).$$

Since

$$\frac{dx}{x}+\frac{dy}{y}=d.\log xy \text{ and } \frac{dx}{x}-\frac{dy}{y}=d.\log\frac{x}{y},$$

we may write also

$$Mdx+Ndy=$$
$$\frac{1}{2}\left\{(Mx+Ny)\,d.\log xy+(Mx-Ny)\,d.\log\frac{x}{y}\right\}. \quad \text{(a)}$$

Now, if $Mx+Ny$ happens to be identically $=0$, we shall have

$$\frac{Mdx+Ndy}{Mx-Ny}=\tfrac{1}{2}\,d.\log\frac{x}{y};$$

and because the second member of this equation is an exact differential, the first member is also one; that is, $\frac{1}{Mx-Ny}$ is an integrating factor.

If, on the contrary, $Mx - Ny$ happens to be identically $= 0$, we shall have

$$\frac{Mdx + Ndy}{Mx + Ny} = \tfrac{1}{2}\, d.\log xy,$$

where, because the second member is an exact differential, the first member is also one; that is, $\frac{1}{Mx + Ny}$ is then an integrating factor.

But, if neither $Mx + Ny$ nor $Mx - Ny$ is identically $=0$, equation (a) divided by $Mx + Ny$ gives

$$\frac{Mdx+Ndy}{Mx+Ny} = \tfrac{1}{2}\, d.\log xy + \frac{1}{2}\,\frac{Mx-Ny}{Mx+Ny}\, d.\log \frac{x}{y}, \quad \text{(b)}$$

and the last term of this equation will be an exact differential, if $\frac{Mx - Ny}{Mx + Ny}$ is a function of $\log \frac{x}{y}$, or generally a function of $\frac{x}{y}$, that is, if it is a homogeneous function of x and y of the degree 0, as is $\varphi\left(\frac{x}{y}\right)$. In such a case, then, the first member of (b) will also be a perfect differential, and the integrating factor will be $\frac{1}{Mx + Ny}$.

Again, if neither $Mx + Ny$ nor $Mx - Ny$ is identically $= 0$, equation (a) divided by $Mx - Ny$ gives

$$\frac{Mdx+Ndy}{Mx-Ny} = \frac{1}{2}\,\frac{Mx+Ny}{Mx-Ny}\, d.\log xy + \tfrac{1}{2}\, d.\log \frac{x}{y} \quad \text{(c)}$$

and the first term of the second member of this equation will be an exact differential if $\frac{Mx + Ny}{Mx - Ny}$ is a function of log xy, or generally of xy, that is, if it is a homogeneous function of x and y of the second degree, as is $\varphi(xy)$. In such a case the first member of (c) will also be an exact differential, the integrating factor being $\frac{1}{Mx - Ny}$.

To give an example of this process of integration, let

$$(x^2 + y^2)\, dx - xydy = 0.$$

Here we have

$$M = x^2 + y^2, \text{ and } N = -xy.$$

Consequently

$$Mx - Ny = x^3 + 2xy^2, \quad Mx + Ny = x^3,$$

and

$$\frac{Mx - Ny}{Mx + Ny} = \frac{x^3 + 2xy^2}{x^3} = 1 + 2\left(\frac{y}{x}\right)^2.$$

Hence, by formula (b),

$$\frac{(x^2 + y^2)\, dx - xydy}{x^3} =$$

$$\tfrac{1}{2}\, d.\log xy + \tfrac{1}{2}\left(1 + 2\,\frac{y^2}{x^2}\right) d.\log \frac{x}{y} = 0.$$

Integrating this expression, we have

$$\tfrac{1}{2}\log xy + \tfrac{1}{2}\log\frac{x}{y} + \int \frac{y^2}{x^2}\, d.\log\frac{x}{y} = C.$$

But, making $\frac{x}{y} = z$, we find

$$\int \frac{y^2}{x^2}\, d.\log \frac{x}{y} =$$

$$\int \frac{d.\log z}{z^2} = \int \frac{dz}{z^3} = -\frac{1}{2z^2} = -\frac{1}{2}\left(\frac{y}{x}\right)^2.$$

And, therefore, the whole integral will reduce to

$$\log x - \frac{1}{2}\left(\frac{y}{x}\right)^2 = C.$$

As the proposed differential was homogeneous, we might have integrated it by assuming $y = zx$, according to the method explained above (No. 83). We would thus have found $dx = zxdz$, whence $\frac{dx}{x} = zdz$, and

$$\log x = \frac{z^2}{2} + C = \frac{1}{2}\left(\frac{y}{x}\right)^2 + C,$$

a result identical with the preceding one, as was to be expected.

Integration of other Differential Equations.

85. We have seen in the preceding pages that our success in the integration of a differential frequently depends on our ability to give it a simpler form. Though we have pointed out many such cases, many others would have to be examined, if we had to give an adequate idea of the resources offered by the Calculus. But, as this is an elementary work, we must content ourselves with giving a few examples of a certain number of other processes frequently adopted by analysts for the transformation of differentials not directly integrable.

I. Let $\quad dy - aydx = f(x)\, dx. \qquad (1)$

If we make $y = uv$, we have $dy = udv + vdu$, and, by substitution,

$$udv + vdu - auvdx = f(x)\,dx.$$

As one of the two quantities u and v can be arbitrarily assumed, take

$$vdu = auvdx, \text{ or } du = audx. \qquad (2)$$

Then the above equation will be reduced to

$$udv = f(x)\,dx. \qquad (3)$$

Now, (2) gives $\log u = ax$, or $u = e^{ax}$; and this, substituted in (3), gives

$$dv = \frac{f(x)\,dx}{e^{ax}}.$$

Whence

$$v = \int \frac{f(x)\,dx}{e^{ax}} + C,\ y = uv = e^{ax}\left(\int \frac{f(x)\,dx}{e^{ax}} + C\right);$$

which is the integral required.

II. Let

$$\frac{d^2y}{dx^2} - a\frac{dy}{dx} + by = 0. \qquad (1)$$

To simplify this expression, assume the two equations

$$\frac{dy}{dx} - ky = z, \text{ and } \frac{dz}{dx} - k'z = 0, \qquad (2)$$

where z is an auxiliary variable, whilst k and k' are constants to be determined. Differentiating the first of equations (2), we have

$$\frac{d^2y}{dx^2} - k\frac{dy}{dx} = \frac{dz}{dx} = k'z,$$

or, since $z = \frac{dy}{dx} - ky$, substituting and reducing,

$$\frac{d^2y}{dx^2} - (k + k')\frac{dy}{dx} + kk'y = 0,$$

which, compared with (1) gives $k + k' = a$, and $kk' = b$.

Now, from the second of equations (2) we have

$$\frac{dz}{z} = k'dx, \text{ and } z = e^{k'x + c}, \text{ or } z = Ce^{k'x};$$

hence, substituting in the first of equations (2), and multiplying by dx,

$$dy - kydx = Ce^{k'x}\,dx,$$

an equation of the same form as the one which we have integrated in the preceding example. We shall have, therefore,

$$y = e^{kx}\int\left(\frac{Ce^{k'x}\,dx}{e^{kx}} + C'\right).$$

But

$$\int\frac{Ce^{k'x}\,dx}{e^{kx}} = \frac{C}{k'-k}\int e^{(k'-k)x}(k'-k)\,dx = \frac{Ce^{(k'-k)x}}{k'-k};$$

therefore

$$y = \frac{e^{kx}}{k'-k}\left(Ce^{(k'-k)x} + C'\right) = C_1e^{k'x} + C_2e^{kx},$$

C_1 and C_2 being two arbitrary constants.

This same integral could be easily obtained by another method which deserves special notice, owing to its simplicity and the range of its application. In the equation

$$\frac{d^2y}{dx^2} - a\frac{dy}{dx} + by = 0 \tag{1}$$

make $y = C_1e^{kx}$. Then, differentiating, we have

$$\frac{dy}{dx} = C_1ke^{kx},\ \frac{d^2y}{dx^2} = C_1k^2e^{kx},$$

and therefore, substituting in (1),

$$C_1e^{kx}(k^2 - ak + b) = 0;$$

or, rejecting the factor, C_1e^{kx}, $k^2 - ak + b = 0$. This *auxiliary equation* gives

$$k = \frac{a + \sqrt{a^2 - 4b}}{2},\ k' = \frac{a - \sqrt{a^2 - 4b}}{2};$$

and these two values of k are connected by the relations $k + k' = a$, and $kk' = b$, as in the preceding solution. We have, therefore, two particular integrals $y = C_1e^{k'x}$, and $y = C_2e^{kx}$, which, added together, will give us for the complete integral *

$$y = C_1e^{k'x} + C_2e^{kx}.$$

If, in (1), we make $a = 3$, $b = 2$, we shall find

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* The *complete* integral of a differential equation of the second order must not only satisfy that equation, but also contain two arbitrary constants (No. 62). The particular integrals $y = C_1e^{k'x}$, and $y = C_2e^{kx}$, satisfy equation (1); for they give

$$C_1e^{k'x}(k'^2 - ak' + b) = 0,$$

and

$$C_2e^{kx}(k^2 - ak + b) = 0;$$

but each contains only a single arbitrary constant. The integral

$$y = C_1e^{k'x} + C_2e^{kx},$$

which is the sum of these particular integrals, contains two arbitrary constants, and equally satisfies equation (1); for it gives

$$(C_1e^{k'x} + C_2e^{kx})(k^2 - ak + b) = 0.$$

Hence $y = C_1e^{k'x} + C_2e^{kx}$ is our complete integral.

$k'=1$, $k=2$; and the complete integral will be $y=C_1e^{2x}+C_2e^{x}$.

III. Let

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+13y=0. \qquad (1)$$

Assuming $y=Ce^{mx}$, differentiating, and substituting in (1), we find

$$Ce^{mx}(m^2-4m+13)=0.$$

Solving the auxiliary equation $m^2-4m+13=0$, we obtain

$$m'=2+3\sqrt{-1}, \quad m''=2-3\sqrt{-1};$$

and the complete integral will be expressed by

$$y=Ce^{2x+3x\sqrt{-1}}+C_1e^{2x-3x\sqrt{-1}}.$$

This integral, by referring to De Moivre's formulas (No. 26), may take the form

$$y=Ce^{2x}.e^{3x\sqrt{-1}}+C_1e^{2x}.e^{-3x\sqrt{-1}}=$$

$$Ce^{2x}(\cos 3x+\sqrt{-1}\sin 3x)$$
$$+C_1e^{2x}(\cos 3x-\sqrt{-1}\sin 3x)=$$

$$(C+C_1)e^{2x}\cos 3x+(C-C_1)\sqrt{-1}\,e^{2x}\sin 3x,$$

or, replacing $C+C_1$ and $(C-C_1)\sqrt{-1}$ by new arbitrary constants,

$$y=Ae^{2x}\cos 3x+Be^{2x}\sin 3x.$$

This is the form of the integral, when the auxiliary equation has two unequal imaginary roots.

IV. Let

$$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-\frac{dy}{dx}+y=0. \qquad (1)$$

Assuming $y = Ce^{mx}$, and differentiating, we find the auxiliary equation

$$m^3 - m^2 - m + 1 = 0,$$

the roots of which are $m_1 = -1$, $m_2 = 1$, $m_3 = 1$.

Owing to the root $m_1 = -1$, we shall have in the integral the term Ce^{-x}. The roots $m_2 = 1$, $m_3 = 1$ will give the terms $C_1e^x + C_2e^x$. But these two terms coalesce into the single term $(C_1 + C_2)e^x$, where $C_1 + C_2$ is equivalent to a single arbitrary constant, whereas the *complete* integral of (1) must contain three arbitrary constants. To remedy the deficiency, let us begin by supposing m_3 to differ from m_2 by a small quantity h. Then

$$C_1e^{m_2x} + C_2e^{m_3x} = C_1e^{m_2x} + C_2e^{(m_2+h)x} = e^{m_2x}(C_1 + C_2e^{hx}),$$

and, developing by the exponential formula (No. 24),

$$C_1e^{m_2x} + C_2e^{m_3x} = e^{m_2x}\left(C_1 + C_2hx + C_2\frac{h^2x^2}{2} + \dots\right)$$

Now, though h must become less than any assignable quantity in order to verify the equation $m_2 = m_3$, yet the product C_2h may remain finite, if we assume C_2 greater than any assignable quantity. Let, then, $C_2h = C_3$ be a new arbitrary constant. The complete integral will be

$$y = Ce^{m_1x} + e^{m_2x}(C_1 + C_3x),$$

and, in our case, with $m_1 = -1$, $m_2 = m_3 = 1$,

$$y = Ce^{-x} + e^x(C_1 + C_3x).$$

V. Let

$$\frac{d^2y}{dx^2} = \frac{2y}{x^2}. \qquad (1)$$

Assume $x = e^\vartheta$, and $y = xz = e^\vartheta z$, ϑ and z being two auxiliary variables. Differentiating the two assumed equations, we have

$$dy = e^\vartheta\, dz + ze^\vartheta\, d\vartheta, \quad dx = e^\vartheta\, d\vartheta,$$

whence

$$\frac{dy}{dx} = \frac{dz}{d\vartheta} + z, \; \frac{d^2y}{dx^2} = e^{-\vartheta}\left(\frac{d^2z}{d\vartheta^2} + \frac{dz}{d\vartheta}\right) = \frac{2y}{x^2}.$$

But

$$\frac{2y}{x^2} = \frac{2e^\vartheta z}{e^{2\vartheta}} = \frac{2z}{e^\vartheta} = e^{-\vartheta}.2z;$$

therefore, substituting, and suppressing the common factor $e^{-\vartheta}$,

$$\frac{d^2z}{d\vartheta^2} + \frac{dz}{d\vartheta} - 2z = 0. \qquad (2)$$

To integrate this, assume $z = Ce^{m\vartheta}$. Then

$$\frac{dz}{d\vartheta} = Cme^{m\vartheta}, \; \frac{d^2z}{d\vartheta^2} = Cm^2e^{m\vartheta},$$

and (2) becomes

$$Ce^{m\vartheta}(m^2 + m - 2) = 0.$$

The auxiliary equation $m^2 + m - 2 = 0$ gives $m_1 = 1$, $m_2 = -2$. Hence

$$z = Ce^\vartheta + C'e^{-2\vartheta}.$$

But $z = \frac{y}{x}$, and $e^\vartheta = x$. Therefore, finally,

$$\frac{y}{x} = Cx + \frac{C'}{x^2}, \text{ or } y = Cx^2 + \frac{C'}{x}.$$

VI. Let

$$\frac{d^2y}{dx^2} \pm a \frac{dy}{dx} + by = mx. \qquad (1)$$

To get rid of the term mx, we shall assume

$$y = z + Px + Q, \qquad (2)$$

P and Q being constants to be suitably determined. From (2) we obtain

$$\frac{dy}{dx} = \frac{dz}{dx} + P, \quad \frac{d^2y}{dx^2} = \frac{d^2z}{dx^2}.$$

These values substituted in (1) give

$$\frac{d^2z}{dx^2} \pm a \frac{dz}{dx} \pm aP + bz + bPx + bQ = mx. \quad (3)$$

Now, let $bP = m$, and $bQ = \mp aP$. Then (3) will be reduced to

$$\frac{d^2z}{dx^2} \pm a \frac{dz}{dx} + bz = 0. \qquad (4)$$

and (2) will become

$$y = z + \frac{m}{b} x \mp \frac{am}{b^2}. \qquad (5)$$

Equation (4) will be easily integrated by the method which we have followed in examples II and III. When z has been found, the integral of (1) will be known by equation (5).

VII. Let

$$\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2y = e^x. \qquad (1)$$

To get rid of the term e^x, we shall assume

$$y = e^x (z + P), \tag{2}$$

P being a constant to be suitably determined. From (2) we obtain

$$\frac{dy}{dx} = e^x (z + P) + e^x \frac{dz}{dx},$$

$$\frac{d^2y}{dx^2} = e^x (z + P) + e^x \frac{dz}{dx} + e^x \frac{dz}{dx} + e^x \frac{d^2z}{dx^2}.$$

Substituting these values in (1), and cancelling the common factor e^x, we have

$$\frac{d^2z}{dx^2} - 2(k-1)\frac{dz}{dx} + (k-1)^2 z + P(k-1)^2 = 1. \tag{3}$$

Hence, if we make

$$P(k-1)^2 = 1, \text{ or } P = \frac{1}{(k-1)^2},$$

we shall have

$$\frac{d^2z}{dx^2} - 2(k-1)\frac{dz}{dx} + (k-1)^2 z = 0,$$

and
$$y = e^x \left(z + \frac{1}{(k-1)^2}\right).$$

If we now make $z = e^{mx}$, the integration of the last differential equation can be made to depend on the auxiliary equation

$$m^2 - 2m(k-1) + (k-1)^2 = 0,$$

which gives $m_1 = k - 1 = m_2$. Hence, by the rule of example IV,

$$z = e^{(k-1)x} (C + C_1 x),$$

and

$$y = e^{x}.e^{(k-1)x}(C + C_1x) + \frac{e^{x}}{(k-1)^2} =$$

$$e^{kx}(C + C_1x) + \frac{e^{x}}{(k-1)^2}.$$

In this example, if $k = 1$, equation (3) reduces to $\frac{d^2z}{dx^2} = 1$, and P disappears. In this case, equations (1) and (2) will become

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{x}, \text{ and } y = e^{x}z.$$

Hence, integrating the equation $\frac{d^2z}{dx^2} = 1$, we shall have for this case,

$$z = \frac{x^2}{2} + Cx + C_1, \text{ and } y = e^{x}\left(\frac{x^2}{2} + Cx + C_1\right).$$

VIII. Let

$$\frac{d^2y}{dx^2} + n^2y = \cos ax. \qquad (1)$$

Assume $y = z + \frac{\cos ax}{h}$, where h is a constant to be determined in view of a future simplification. We shall have

$$\frac{dy}{dx} = \frac{dz}{dx} - \frac{a}{h}\sin ax,$$

$$\frac{d^2y}{dx^2} = \frac{d^2z}{dx^2} - \frac{a^2}{h}\cos ax.$$

Thus (1), by substitution, will become

$$\frac{d^2z}{dx^2} - \frac{a^2}{h} \cos ax + n^2z + \frac{n^2}{h} \cos ax = \cos ax,$$

or

$$\frac{d^2z}{dx^2} + n^2z = \cos ax - \frac{n^2 - a^2}{h} \cos ax.$$

Take $h = n^2 - a^2$, and the equation will reduce to

$$\frac{d^2z}{dx^2} + n^2z = 0. \qquad (2)$$

To integrate this, assume $z = e^{mx}$, from which we find, by differentiation,

$$m^2e^{mx} + n^2e^{mx} = 0, \text{ or } m^2 + n^2 = 0,$$

and therefore, $m = \pm n\sqrt{-1}$. Hence, by the method followed in example III,

$$z = Ce^{nx\sqrt{-1}} + C_1e^{-nx\sqrt{-1}},$$

that is,

$$z = A \cos nx + B \sin nx,$$

and

$$y = A \cos nx + B \sin nx + \frac{\cos ax}{n^2 - a^2},$$

A and B being arbitrary constants.

Integration by Elimination of Differentials.

86. There are differentials of which the integral can be found without direct integration. They are those whose form admits of a ready elimination of the differentials themselves. We shall show by a few examples how this method can be utilized.

I. Let

$$x = a + m\frac{dy}{dx} + n\left(\frac{dy}{dx}\right)^3. \qquad (1)$$

Making $\frac{dy}{dx} = p$, whence $dy = pdx$, the equation becomes

$$x = a + mp + np^3. \qquad (2)$$

Differentiating (2), we have $dx = mdp + 3np^2dp$, whence

$$pdx, \text{ or } dy = mpdp + 3np^3dp,$$

and

$$y = \tfrac{1}{2}mp^2 + \tfrac{3}{4}np^4 + C. \qquad (3)$$

If we can eliminate p between (2) and (3), we shall have the expression of the complete integral.

II. Let

$$y = m\left(\frac{dy}{dx}\right)^2 + 2n\left(\frac{dy}{dx}\right)^3. \qquad (1)$$

Making $\frac{dy}{dx} = p$, the equation becomes

$$y = mp^2 + 2np^3. \qquad (2)$$

Differentiating (2), we have

$$dy = 2mpdp + 6np^2dp = pdx;$$

whence

$$dx = 2mdp + 6npdp,$$

and therefore

$$x = 2mp + 3np^2 + C. \qquad (3)$$

From this equation we have

$$p = \frac{-m \pm \sqrt{3nx + m^2 - 3nC}}{3n}.$$

This value of p being placed in (2), we shall have the complete integral of the proposed equation.

III. Let

$$x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + m = 0. \qquad (1)$$

Dividing by $\frac{dy}{dx} = p$, we obtain

$$y = xp + \frac{m}{p}. \qquad (2)$$

Differentiating, dividing by dx, and reducing, we have

$$\left(x - \frac{m}{p^2}\right)\frac{dp}{dx} = 0. \qquad (3)$$

This equation must be satisfied either by $\frac{dp}{dx} = 0$, or by $x - \frac{m}{p^2} = 0$. If $\frac{dp}{dx} = 0$, then $p = C$, and (2) becomes

$$y = Cx + \frac{m}{C}. \qquad (4)$$

If $x - \frac{m}{p^2} = 0$, then $p = \sqrt{\frac{m}{x}}$, which value substituted in (2) will give

$$y^2 = 4mx. \qquad (5)$$

an integral without any arbitrary constant, that is, *a singular solution;** whereas equation (4) gives

* By *singular solutions* we mean such results as are not contained in, or cannot be obtained from, the complete integrals, whatever particular value be assigned to the arbitrary constant. They can only be obtained by differentiating the complete integrals *with respect to the constant alone*, considering it as a *variable parameter*, as we have done above (Nos. 49, 50), and substituting its value, in terms of the vari-

the complete integral, and represents the tangents by whose intersections the parabola $y^2 = 4mx$ is generated.

IV. Let

$$x + y\frac{dy}{dx} - a\left(\frac{dy}{dx}\right)^2 = 0. \qquad (1)$$

We write, as usual,

$$x + py - ap^2 = 0 \qquad (2)$$

whence

$$dx + ydp + pdy - 2apdp = 0. \qquad (3)$$

From (2) we have $y = \dfrac{ap^2 - x}{p}$. Substituting this in (3) and reducing, we find

$$dx(1 + p^2) - x\frac{dp}{p} = apdp,$$

which, being divided by $p\sqrt{1+p^2}$, will take the form

$$\frac{pdx\sqrt{1+p^2} - \dfrac{xdp}{\sqrt{1+p^2}}}{p^2} = \frac{adp}{\sqrt{1+p^2}};$$

the integral of which is

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ables, in the expression of the integrals. Thus, differentiating (4) with regard to C alone, we find $C = \sqrt{\dfrac{m}{x}}$, a value which will reduce (4) to $y^2 = 4mx$.

It is not our purpose to discuss the relations existing between the complete integrals and the singular solutions. We will simply state that, owing to such relations, as interpreted by Geometry, the latter are often called *envelopes* of the former. This subject has been very deeply treated in the excellent work of Mr. George Boole on *Differential Equations*.

$$\frac{x\sqrt{1+p^2}}{p} = a \log\left(p + \sqrt{1+p^2}\right) + C,$$

or

$$x = \frac{p}{\sqrt{1+p^2}}\left(C + a \log\left(p + \sqrt{1+p^2}\right)\right). \quad (4)$$

Now, from (2) we have

$$p = \frac{y \pm \sqrt{y^2 + 4ax}}{2a}.$$

This value of p, placed in (4), will give the complete integral.

V. Let

$$ydx - xdy = n\sqrt{dx^2 + dy^2}. \quad (1)$$

Dividing by dx, and making $\frac{dy}{dx} = p$, as usual, we shall have

$$y = px + n\sqrt{1+p^2}, \quad (2)$$

whence

$$dy = pdx + xdp + \frac{npdp}{\sqrt{1+p^2}},$$

or, since $dy = pdx$,

$$dp\left(x + \frac{np}{\sqrt{1+p^2}}\right) = 0. \quad (3)$$

This equation must be satisfied either by $dp = 0$, or by $x + \frac{np}{\sqrt{1+p^2}} = 0$. If $dp = 0$, then $p = C$; and (2) becomes

$$y = Cx + n\sqrt{1 + C^2},$$

which is the complete integral. If $x+\frac{np}{\sqrt{1+p^2}}=0$, then $p=-\frac{x}{\sqrt{n^2-x^2}}$; which value placed in (2) gives

$$y^2+x^2=n^2,$$

an integral without any arbitrary constant, representing a *singular solution.* This solution is the locus of the intersections of all the straight lines obtained by varying the constant C in the complete integral.

Double Integrals.

87. When an infinitesimal area is referred to rectangular co-ordinates, its most general expression is the infinitely small rectangle of which dx and dy are the sides; and thus $dA = dxdy$.

To find the area A, we must integrate between proper limits with respect to each variable in succession. This double integration is indicated by writing the sign of integration twice before the quantity to be integrated. Thus we have the *double integral*

$$A=\iint dydx.$$

One of these integrations can always be performed, so that we may have either

$$\int ydx+C, \text{ or } \int xdy+C.$$

Then, if the equation of the line that bounds the area be $y=f(x)$, or $x=\varphi(y)$, we shall have either

$$\int f(x)\,dx + C, \text{ or } \int \varphi(y)\,dy + C,$$

that is, a function of a single variable, to be integrated within proper limits, as usual.

In like manner, an infinitesimal volume may be expressed by the product of its infinitesimal dimensions dx, dy, dz. Hence

$$dV = dxdydz, \text{ and } V = \iiint dxdydz,$$

a *triple integral*, to be taken between proper limits with respect to each variable in succession. The order of the integrations is wholly arbitrary. If it is found convenient to integrate first with regard to z, the integral, between $z = 0$ and $z = z$, will be

$$V = \iint dxdy\,.\,z,$$

and if $z = f(x, y)$ be the equation of the surface that bounds the volume, then we write

$$V = \iint f(x, y)\,dxdy,$$

and we proceed to the double integration, as we have just explained. Some examples of this method of integration will be given in the following section.

SECTION II.

APPLICATION OF INTEGRAL CALCULUS TO GEOMETRY.

88. Our object in this section is to find definite values for the length of some curves, for their areas, surfaces of revolution, and volumes of revolution. The operation by which we determine the length of a curve, is called *rectification;* no curve, however, is properly said to be *rectified*, unless its length be expressed by a finite number of algebraic terms, to the exclusion of transcendentals. The operation by which we determine areas and surfaces, is called *quadrature;* and the operation by which volumes are determined, is called *cubature.*

Rectification of Curves.

89. The arc s of a curve referred to rectangular axes will be found by integrating between proper limits the formula (No. 51),

$$ds = \sqrt{dx^2 + dy^2}.$$

EXAMPLE I. Let the curve be a parabola $y^2 = 2px$. Then $dx = \frac{ydy}{p}$. Placing this value of dx in the formula, we have

$$ds = \sqrt{\frac{y^2dy^2}{p^2} + dy^2} = \frac{dy}{p}\sqrt{y^2 + p^2}.$$

and integrating by formulas (34) and (28),

$$s = \frac{y\sqrt{y^2+p^2}}{2p} + \frac{p}{2}\log\left(y + \sqrt{y^2+p^2}\right) + C,$$

or, taking the integral between $y=0$ and $y=y$,

$$s = \frac{y\sqrt{y^2+p^2}}{2p} + \frac{p}{2}\log\left(\frac{y+\sqrt{y^2+p^2}}{p}\right).$$

EXAMPLE II. Let the curve be a cycloid. Then (No. 48), $dx = \frac{ydy}{\sqrt{2ry - y^2}}$. Placing this value of dx in our formula, and reducing, we have

$$ds = dy\sqrt{\frac{2ry}{2ry - y^2}} = \sqrt{2r}\,\frac{dy}{\sqrt{2r - y}}.$$

Hence

$$s = -\sqrt{2r}.2\sqrt{2r - y} + C;$$

and taking the integral from $y = 0$ to $y = y$,

$$s = \sqrt{2r}\,(-2\sqrt{2r - y} + 2\sqrt{2r}).$$

Make $y = 2r$; then $s = 4r =$ one-half of the cycloidal curve.

EXAMPLE III. Let the curve be a circle. Its equation $x^2 + y^2 = r^2$ gives $dy = -\frac{xdx}{\sqrt{r^2 - x^2}}$. Substituting this value in our formula, we get

$$ds = dx\sqrt{1 + \frac{x^2}{r^2 - x^2}} = r\,\frac{dx}{\sqrt{r^2 - x^2}}.$$

Make $r = 1$. Then

$$s = \int dx\,(1 - x^2)^{-\frac{1}{2}} =$$

$$\int dx\left(1 + \frac{x^2}{2} + \frac{3}{2.4}x^4 + \frac{3.5}{2.4.6}x^6 + \ldots\right),$$

and integrating from $x = 0$ to $x = x$,

$$s = x + \frac{1}{2.3}x^3 + \frac{1.3}{2.4.5}x^5 + \frac{1.3.5}{2.4.6.7}x^7 + \ldots$$

90. Let $s = \tan^{-1} x$ be a circular arc. Its differential will be

$$ds = \frac{dx}{1 + x^2},$$

or developing the second member of this expression

$$ds = dx\,(1 - x^2 + x^4 - x^6 + \ldots)$$

This, integrated from $x = 0$ to $x = x$, will give

$$s = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

When $x = 1$, then $s = \frac{\pi}{4}$. Hence this series may serve to calculate the value of π. However, as it converges but slowly for $x = 1$, the calculation has been facilitated by the following artifice:

By trigonometry we have

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

Make $\qquad a = \tan^{-1} m,\ b = \tan^{-1} n,$

and $\qquad (a + b) = \tan^{-1} z = \tan^{-1} m + \tan^{-1} n.$

Then

$$z = \tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

But $\tan a = m$, and $\tan b = n$; and therefore

$$z = \frac{m + n}{1 - mn}, \text{ whence } n = \frac{z - m}{1 + mz}.$$

Now, assuming $z = 1$, $m = \frac{1}{5}$, we find $n = \frac{2}{3}$. Therefore

$$\tan^{-1} 1 = \tan^{-1} \tfrac{1}{5} + \tan^{-1} \tfrac{2}{3}.$$

Assuming $z = \frac{2}{3}$, $m = \frac{1}{5}$, we find $n = \frac{7}{17}$. Therefore

$$\tan^{-1} \tfrac{2}{3} = \tan^{-1} \tfrac{1}{5} + \tan^{-1} \tfrac{7}{17}.$$

Assuming $z = \frac{7}{17}$, $m = \frac{1}{5}$, we find $n = \frac{9}{46}$. Therefore

$$\tan^{-1} \tfrac{7}{17} = \tan^{-1} \tfrac{1}{5} + \tan^{-1} \tfrac{9}{46}.$$

Assuming $z = \frac{9}{46}$, $m = \frac{1}{5}$, we find $n = -\frac{1}{239}$. Therefore

$$\tan^{-1} \tfrac{9}{46} = \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239}.$$

Hence, by successive substitutions, we at last find

$$\tan^{-1} = \frac{\pi}{4} = 4 \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239}.$$

Thus, if we make $x = \frac{1}{5}$ in the formula, we find the value of $\tan^{-1} \frac{1}{5}$, and in like manner, if we make $x = \frac{1}{239}$ we find the value of $\tan^{-1} \frac{1}{239}$. The calculation of six terms for $\tan^{-1} \frac{1}{5}$, and of two terms for $\tan^{-1} \frac{1}{239}$, will give the value of $\frac{\pi}{4}$ up to the tenth decimal.

91. The differential of an arc, when expressed in function of polar co-ordinates, is (No. 52)

$$ds = \sqrt{d\rho^2 + \rho^2 d\varphi^2},$$

ρ being the radius vector, and φ the angle which ρ makes with the initial line. The integration will be made as follows.

EXAMPLE I. *To find an arc of the spiral of Archimedes.* The equation of the curve is $\rho = a\varphi$, where $a = \frac{r}{2\pi}$. Hence $d\varphi = \frac{d\rho}{a}$, and

$$ds = \frac{1}{a} d\rho \sqrt{a^2 + \rho^2}.$$

Integrating by parts,

$$\int d\rho \sqrt{a^2 + \rho^2} = \rho \sqrt{a^2 + \rho^2} - \int \frac{\rho^2 d\rho}{\sqrt{a^2 + \rho^2}};$$

and the last term, integrated by formula (33), gives

$$\int \frac{\rho^2 d\rho}{\sqrt{a^2 + \rho^2}} = \tfrac{1}{2} \rho \sqrt{a^2 + \rho^2} - \frac{a^2}{2} \log (\rho + \sqrt{\rho^2 + a^2}).$$

Hence

$$s = \frac{1}{2a} \rho \sqrt{a^2 + \rho^2} + \frac{a}{2} \log (\rho + \sqrt{\rho^2 + a^2}) + C;$$

and taking the integral from $\rho = 0$ to $\rho = r$, we have for the first spire

$$s = \frac{1}{2a} r \sqrt{a^2 + r^2} + \frac{a}{2} \log \frac{r + \sqrt{a^2 + r^2}}{a}$$

or, since $a = \frac{r}{2\pi}$,

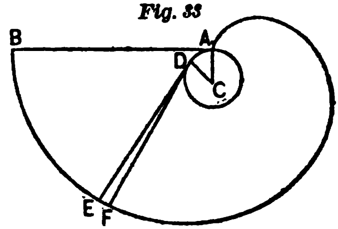
$$s = \frac{r}{2} \sqrt{1 + 4\pi^2} + \frac{r}{4\pi} \log (2\pi + \sqrt{1 + 4\pi^2}).$$

To obtain the length of n spires, it suffices to take the integral from $\rho = 0$ to $\rho = nr$.

EXAMPLE II. To find the length of the curve traced by the end B of a tense string AB (Fig. 33), whose other end A is fixed on a circle around which the string is being wrapped.

Let $AB = l$, and $AC = r$, and let E be a point on the curve. Draw ED tangent to the circle, and draw the radius CD. Make $ACD = \vartheta$, and consequently, the arc $AD = r\vartheta$. Then the infinitesimal arc $ds = EF$ described by ED will be equal to $ED \times d\vartheta$. But, by the nature of the case, $ED = AB - AD = l - r\vartheta$. Therefore

Fig. 33



$$ds = (l - r\vartheta)\,d\vartheta.$$

Integrating from $\vartheta = 0$ to $\vartheta = \vartheta$, we have

$$s = l\vartheta - \tfrac{1}{2}r\vartheta^2.$$

As a particular case, assume $l = 2\pi r$; then the whole curve will be described when $\vartheta = 2\pi$. Then $s = 2\pi^2 r$, or $s = \frac{l^2}{2r}$.

Quadrature of Curves.

92. The differential of the area of a curve referred to rectangular axes is (No. 51)

$$dA = ydx.$$

This equation, integrated between proper limits, will give us the area intercepted between the curve, the axis of x, and the limiting ordinates.

Example I. Let the curve be a parabola $y^2 = 2px$. Then $dx = \frac{ydy}{p}$; whence

$$dA = \frac{y^2dy}{p}$$

Integrating from $y=0$ to $y=y$, we have

$$A=-\frac{1}{p}\int_0^y y^2dy=\frac{y^3}{3p}=\frac{2xy}{3}.$$

EXAMPLE II. Let the curve be a cycloid. Then $dx=\frac{ydy}{\sqrt{2ry-y^2}}$ whence

$$dA=\frac{y^2dy}{\sqrt{2ry-y^2}}.$$

Integrating by formula (37), we have

$$\int\frac{y^2dy}{\sqrt{2ry-y^2}}=-\frac{y\sqrt{2ry-y^2}}{2}+\frac{3r}{2}\int\frac{ydy}{\sqrt{2ry-y^2}};$$

and again, by the same formula,

$$\int\frac{ydy}{\sqrt{2ry-y^2}}=-\sqrt{2ry-y^2}+r\ \text{vers}^{-1}\frac{y}{r}.$$

Substituting this value in the last term of the preceding equation, and taking the integral from $y=0$ to $y=2r$, we have

$$A=\frac{3r^2}{2}\ \text{vers}^{-1}\frac{2r}{r}=\frac{3\pi r^2}{2}.$$

This is the area of a semi-cycloid. The area of the whole cycloid is therefore $3\pi r^2$, or three times the area of the generating circle.

EXAMPLE III. Let the curve be a circle. In this case, since $y=\sqrt{r^2-x^2}$, we have

$$dA=dx\sqrt{r^2-x^2}$$

whence (No. 73)

$$A=\frac{x}{2}\sqrt{r^2-x^2}+\frac{r^2}{2}\sin^{-1}\frac{x}{r}+C.$$

Taking this integral from $x=0$ to $x=r$, we shall have

$$A = \tfrac{1}{4}\pi r^2,$$

for a quadrant, and therefore $4A = \pi r^2$ for the whole circle.

EXAMPLE IV. Let the curve be the logarithmic $y = \log x$. Then

$$dA = \log x dx.$$

Integrating by parts, we have

$$\int \log x dx = x \log x - \int dx = x(\log x - 1) + C.$$

Taking this integral from $x = 1$ to $x = x$, we have

$$A = x \log x - x + 1$$

for the area extending above the axis of x, so long as we take $x > 1$. But if we take the integral from $x = 0$ to $x = 1$, then we find

$$A = 1 - [x(\log x - 1)]_0 = 1;$$

for we know (No. 37) that $x \log x - x = 0$ when $x = 0$. Thus $A = 1$ is the area extending to infinity beneath the axis of x from $x = 1$ to $x = 0$, that is, from $y = 0$ to $y = -\infty$.

EXAMPLE V. Let the curve be the equilateral hyperbola $xy = m^2$. Then $y = \frac{m^2}{x}$; hence

$$dA = m^2 \frac{dx}{x},$$

and integrating from $x = 1$ to $x = x$,

$$A = m^2 \log x.$$

When $m=1$, then $A=\log x$. Thus the Napierian logarithms, whose modulus is $=1$, exhibit so many areas taken in the hyperbola $xy=1$. Hence they are also called *hyperbolic* logarithms.

93. When the curve is referred to polar co-ordinates, then (No 52)

$$dA=\tfrac{1}{2}\rho^2 d\varphi.$$

In the *spiral of Archimedes*, in which $\rho=a\varphi$, and $d\rho=ad\varphi$, we have

$$dA=\tfrac{1}{2}a^2\varphi^2 d\varphi, \text{ or } dA=\frac{1}{2a}\rho^2 d\rho.$$

Integrating the first expression from $\varphi=0$ to $\varphi=2\pi$, and the second from $\rho=0$ to $\rho=r$, we have

$$A=\frac{a^2}{6}(2\pi)^3, \text{ and } A=\frac{1}{6}\cdot\frac{r^3}{a};$$

and, as $a=\frac{r}{2\pi}$, both expressions reduce to $A=\frac{1}{3}\pi r^2$.

In the *parabolic spiral*, in which $\rho^2=2a\varphi$, and $\rho d\rho=ad\varphi$, we have

$$dA=a\varphi d\varphi,$$

and integrating from $\varphi=0$ to $\varphi=2\pi$,

$$A=2a\pi^2.$$

But, as $a=\frac{r^2}{4\pi}$ (No. 55), this surface becomes $A=\frac{\pi r^2}{2}$.

In the *hyperbolic spiral*, in which $\rho\varphi = a$, and $d\varphi = -\frac{a d\rho}{\rho^2}$, we have

$$dA = -\tfrac{1}{2}\rho^2 \frac{a d\rho}{\rho^2} = -\frac{a}{2} d\rho.$$

And integrating from $\rho = \rho$ to $\rho = a$,

$$A = \frac{a}{2} (\rho - a),$$

a being the radius of the measuring circle.

In the *logarithmic spiral*, in which $\varphi = a \log \rho$, and $d\varphi = \frac{a d\rho}{\rho}$, we have

$$dA = \frac{a}{2} \rho d\rho,$$

and integrating from $\rho = 1$ to $\rho = \rho$,

$$A = \frac{a}{4} (\rho^2 - 1).$$

Surfaces of Revolution.

94. The area of a surface of revolution will be found by integrating the differential expression (No. 51)

$$dS = 2\pi y \sqrt{dx^2 + dy^2}.$$

EXAMPLE I. *Surface of a cone.* The convex surface of a cone is generated by a straight line $y = ax$ revolving about the axis of x. We have, in this case, $dy = adx$; hence

$$dS = 2\pi a x dx \sqrt{a^2 + 1};$$

and integrating from $x = 0$ to $x = h$,

$$S = \pi a h^2 \sqrt{a^2 + 1}.$$

Let r be the radius of the base; then $a = \frac{r}{h}$; and $S = 2\pi r . \frac{\sqrt{h^2 + r^2}}{2}$; which is the common expression of S in Geometry.

EXAMPLE II. *Surface of a sphere.* From the equation $y = \sqrt{2rx - x^2}$ of the circle, we have $dy = \frac{1}{y}(r - x) dx$; hence

$$dS = 2\pi y \sqrt{dx^2 + \frac{(r-x)^2}{y^2} dx^2} = 2\pi r dx;$$

and integrating from $x = 0$ to $x = 2r$,

$$S = 4\pi r^2,$$

which is the surface of the sphere.

EXAMPLE III. *Surface of a paraboloid of revolution.* From the equation $y^2 = 2px$ of the parabola, we have $dy = \frac{pdx}{y}$; hence

$$dS = 2\pi \sqrt{2px} \sqrt{\left(1 + \frac{p}{2x}\right)} dx^2 = 2\pi dx \sqrt{2px + p^2};$$

and integrating from $x = 0$ to $x = x$,

$$S = \frac{2\pi}{3p} \left(\sqrt{(p^2 + 2px)^3} - p^3 \right).$$

EXAMPLE IV. ***Surface generated by a cycloid revolving about its base.*** In the cycloid we have

$$dx = \frac{ydy}{\sqrt{2ry - y^2}};$$

hence, by substitution and reduction,

$$dS = 2\pi\sqrt{2r}\,\frac{ydy}{\sqrt{2r - y}},$$

or, making $2r - y = z^2$, whence $y = 2r - z^2$, and $dy = -2zdz$,

$$dS = -4\pi\sqrt{2r}\,(2r - z^2).dz,$$

and

$$S = -4\pi\sqrt{2r}\left(2rz - \frac{z^3}{3}\right),$$

or

$$S = -4\pi\sqrt{2r}\,\left(2r\sqrt{2r - y} - \tfrac{1}{3}\sqrt{(2r - y)^3}\right) + C.$$

Taking the integral from $y = 0$ to $y = 2r$, and reducing, we get

$$S = \tfrac{32}{3}\pi r^2.$$

Hence the whole surface generated will be $= \tfrac{64}{3}\pi r^2$.

Solids of Revolution.

95. The volume of a solid of revolution will be found by integrating between proper limits the differential (No. 51)

$$dV = \pi y^2 dx.$$

EXAMPLE I. *Volume of the frustum of a cone.* The cone is generated by the revolution of a line

$y = ax$ about the axis of x. Hence we have $y^2 = a^2x^2$. Substituting this value of y^2, and integrating from $x = h'$ to $x = h$, we find for the volume of the frustum

$$V = \pi \frac{a^2}{3} (h^3 - h'^3) = \pi a^2 \frac{h - h'}{3} (h^2 + hh' + h'^2),$$

where $h - h'$ is the altitude of the frustum, $\pi a^2 h^2$ its lower base, $\pi a^2 h'^2$ its upper base, and $\pi a^2 hh'$ a mean proportional between the two bases, according to a well-known theorem of Geometry. Making $h' = 0$, we find for the volume of the whole cone, $V = \pi y^2 \frac{h}{3}$.

EXAMPLE II. *Volume of the sphere.* The equation of the generating circle being $y^2 = r^2 - x^2$, our differential formula becomes

$$dV = \pi (r^2 - x^2) dx;$$

whence, integrating from $x = -r$ to $x = r$,

$$V = \pi \left(2r^3 - \frac{2r^3}{3}\right) = \frac{4\pi r^3}{3}.$$

EXAMPLE III. *Volume of the prolate spheroid,* which is generated by the revolution of an ellipse about its transverse axis. We have $y^2 = \frac{b^2}{a^2} (a^2 - x^2)$; hence

$$dV = \pi \frac{b^2}{a^2} (a^2 - x^2) dx,$$

and integrating from $x = -a$ to $x = a$,

$$V = \frac{4\pi b^2 a}{3}.$$

EXAMPLE IV. *Volume of the oblate spheroid*, which is generated by the revolution of an ellipse about its conjugate axis. In this case, the differential of the volume is $dV = \pi x^2 dy$. Now

$$x^2 = \frac{a^2}{b^2}(b^2 - y^2);$$

hence

$$dV = \pi\frac{a^2}{b^2}(b^2 - y^2)\,dy,$$

and integrating from $y = -b$ to $y = b$,

$$V = \frac{4\pi a^2 b}{3}.$$

EXAMPLE V. *Volume of the paraboloid of revolution.* The equation of the revolving parabola being $y^2 = 2px$, we shall have

$$dV = 2\pi pxdx,$$

and integrating from $x = 0$ to $x = x$,

$$V = \pi px^2 = \pi y^2 . \frac{x}{2},$$

which is equal to the volume of a cylinder having the same base, and half the height of the paraboloid.

EXAMPLE VI. *Volume generated by the revolution of a cycloid about its base.* Since, in this case,

$$dx = \frac{ydy}{\sqrt{2ry - y^2}},$$

the differential becomes

$$dV = \frac{\pi y^3 dy}{\sqrt{2ry - y^2}}.$$

By formula (37) applied three times in succession, we shall find

$$V = \pi \left\{ -\frac{2y^2 + 5ry + 15r^2}{6} \sqrt{2ry - y^2} + \frac{5r^3}{2} \text{vers}^{-1} \frac{y}{r} \right\} + C.$$

Taking the integral from $y = 0$ to $y = 2r$, we have for one-half of the volume

$$V = \tfrac{5}{2}\pi^2 r^3;$$

hence the whole volume will be $= 5\pi r^2 \times \pi r$.

Other Geometrical Problems.

96. PROBLEM I. *To find the curve whose subtangent is constant.*

The expression for the subtangent is (No. 34) $\frac{ydx}{dy}$.

Hence if a be the constant,

$$\frac{ydx}{dy} = a.$$

Consequently,

$$\frac{dx}{a} = \frac{dy}{y}, \text{ and } x = a \log y + C.$$

This equation shows that the curve is a logarithmic.

PROBLEM II. *To find the curve whose subnormal is constant.*

The expression for the subnormal is (No. 35) $\frac{ydy}{dx}$.

Hence if a be the constant,

$$\frac{ydy}{dx} = a.$$

Consequently,

$$ydy = adx, \text{ and } y^2 = 2ax + C.$$

The curve is therefore a parabola whose parameter is $2a$, and whose vertex lies anywhere on the axis of x.

Problem III. *To find the curve whose normal is constant.*

The expression for the normal is

$$y\frac{ds}{dx}, \text{ or } y\sqrt{1 + \frac{dy^2}{dx^2}}.$$

Hence, if a be the constant,

$$y\sqrt{1 + \frac{dy^2}{dx^2}} = a, \text{ whence } dx = \pm\frac{ydy}{\sqrt{a^2 - y^2}},$$

and integrating,

$$x = \pm\sqrt{a^2 - y^2} + C,$$

or

$$(x - C)^2 + y^2 = a^2.$$

The curve is therefore a circle having the radius a.

Problem IV. *To find the curve whose tangent is constant.*

The expression for the tangent is

$$y\frac{ds}{dy}, \text{ or } y\sqrt{1 + \frac{dx^2}{dy^2}}.$$

Hence if a be the constant,

$$y\sqrt{1+\frac{dx^2}{dy^2}}=a,\ \text{whence}\ dx=\pm\frac{dy}{y}\sqrt{a^2-y^2}.$$

Multiplying and dividing by $\sqrt{a^2-y^2}$, we find

$$dx=\pm\frac{dy}{y}\cdot\frac{a^2-y^2}{\sqrt{a^2-y^2}}=\mp\frac{ydy}{\sqrt{a^2-y^2}}\pm\frac{a^2dy}{y\sqrt{a^2-y^2}},$$

and

$$x=\pm\sqrt{a^2-y^2}\pm a^2\int\frac{dy}{y\sqrt{a^2-y^2}}.$$

Make $a^2-y^2=z^2$, whence $dy=-\dfrac{zdz}{\sqrt{a^2-z^2}}$; then

$$\int\frac{dy}{y\sqrt{a^2-y^2}}=-\int\frac{dz}{a^2-z^2}=$$

$$-\frac{1}{2a}\int\left(\frac{dz}{a+z}+\frac{dz}{a-z}\right)=-\frac{1}{2a}\log\frac{a+z}{a-z}.$$

But

$$\frac{1}{2a}\log\frac{a+z}{a-z}=\frac{1}{2a}\log\frac{(a+z)^2}{a^2-z^2}=$$

$$\frac{1}{a}\log\frac{a+z}{\sqrt{a^2-z^2}}=\frac{1}{a}\log\frac{a+\sqrt{a^2-y^2}}{y}.$$

And therefore

$$a^2\int\frac{dy}{y\sqrt{a^2-y^2}}=-a\log\frac{a+\sqrt{a^2-y^2}}{y}.$$

Consequently

$$x = \pm \sqrt{a^2 - y^2} \mp a \log \frac{a + \sqrt{a^2 - y^2}}{y} + C.$$

The curve of this equation is called the *tractrix*.

PROBLEM V. *To find the curve in which the square of the arc is proportional to the ordinate.*

The equation of condition will evidently be $s^2 = ay$, a being a constant. Differentiating, we have

$$2sds = ady,$$

$$ds = \frac{ady}{2s} = \frac{ady}{2\sqrt{ay}} = \sqrt{dx^2 + dy^2},$$

whence

$$dx = dy \sqrt{\frac{a}{4y} - 1}.$$

Making $a = 8r$, we shall have

$$dx = dy \sqrt{\frac{2r}{y} - 1} =$$

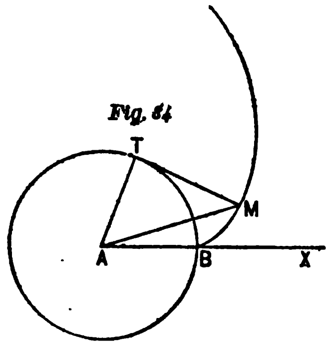
$$dy \sqrt{\frac{2ry - y^2}{y^2}} = \frac{dy}{y} \sqrt{2ry - y^2},$$

whence

$$\frac{dy}{dx} = \frac{y}{\sqrt{2ry - y^2}}.$$

Now, this equation differs from the differential equation of the cycloid in this only, that we have $\frac{dy}{dx}$ instead of $\frac{dx}{dy}$. It follows that our equation belongs to a cycloid *whose vertex has the axis of* x *for tangent*. Hence the curve is a cycloid generated by a circle whose radius is $r = \frac{a}{8}$.

PROBLEM VI. *To find the curve whose evolute is a circle.*



Let $AB = r$ (Fig. 34) be the radius of a circle, and let the curve start from the point B. Let M be a point in the curve, whose co-ordinates are x' and y'. Draw MT tangent to the circle, and therefore normal to the curve; then MT will be the radius of curvature at the point M, as is plain from the theory of evolutes (No. 47).

Now, the equation of the normal is (No. 35)

$$y - y' = -\frac{dx'}{dy'}(x - x'),$$

and since the line passes through the point T, whose co-ordinates we shall designate by α and β, assuming $x = \alpha$ and $y = \beta$, we shall have

$$\beta - y' = -\frac{dx'}{dy'}(\alpha - x'). \qquad (1)$$

But, as MT is tangent to the circle at the point (α, β), we have also from the general equation of the tangent (No. 34)

$$\beta - y' = \frac{d\beta}{d\alpha}(\alpha - x').$$

Therefore

$$\frac{dx'}{dy'} = -\frac{d\beta}{d\alpha}.$$

Now, since $\alpha^2+\beta^2=r^2$, therefore $\alpha d\alpha+\beta d\beta=0$, and $\frac{d\beta}{d\alpha}=-\frac{\alpha}{\beta}$; and consequently $\frac{dx'}{dy'}=\frac{\alpha}{\beta}$, or, omitting the accents, and clearing of fractions,

$$\alpha dy-\beta dx=0. \qquad (2)$$

On the other hand, equation (1) will become

$$\beta-y=-\frac{\alpha}{\beta}(\alpha-x),$$

and this, owing to the relation $\alpha^2+\beta^2=r^2$, reduces to

$$\alpha x+\beta y=r^2. \qquad (3)$$

From (2) and (3) we obtain

$$\alpha=\frac{r^2dx}{xdx+ydy}, \qquad \beta=\frac{r^2dy}{xdx+ydy}.$$

Squaring these equations, adding them together, and changing $\alpha^2+\beta^2$ into r^2, we find

$$(xdx+ydy)^2=r^2(dx^2+dy^2).$$

Let ρ be the radius vector AM. Since $\rho^2=x^2+y^2$, we shall have $\rho d\rho=xdx+ydy$; also, if $BM=s$, we shall have $dx^2+dy^2=ds^2$. Therefore, substituting, and extracting the square root,

$$\rho d\rho=rds \qquad (4)$$

whence

$$\rho^2=2rs+C.$$

When $s=0$, then $\rho=r$. Therefore $C=r^2$; and accordingly

$$\rho^2=2rs+r^2.$$

Let the angle $MAB = \varphi$; then $ds = \sqrt{d\rho^2 + \rho^2 d\varphi^2}$; and (4) becomes

$$\rho d\rho = r\sqrt{d\rho^2 + \rho^2 d\varphi^2},$$

whence

$$r d\varphi = \frac{d\rho}{\rho}\sqrt{\rho^2 - r^2};$$

and multiplying and dividing this by $\sqrt{\rho^2 - r^2}$, and integrating,

$$\varphi = \frac{1}{r}\sqrt{\rho^2 - r^2} + \sin^{-1}\frac{r}{\rho} + C.$$

When $\varphi = 0$, we have $\rho = r$; hence $C = -\frac{\pi}{2}$; and therefore

$$\varphi = \frac{1}{r}\sqrt{\rho^2 - r^2} - \cos^{-1}\frac{r}{\rho}. \qquad (5)$$

Such is the polar equation of the curve required.

To find a corresponding equation with rectangular co-ordinates, we may remark that equation (5) gives

$$\frac{r}{\rho} = \cos\left(\frac{1}{r}\sqrt{\rho^2 - r^2} - \varphi\right),$$

or

$$\frac{r}{\rho} = \cos\varphi \cos\frac{1}{r}\sqrt{\rho^2 - r^2} + \sin\varphi \sin\frac{1}{r}\sqrt{\rho^2 - r^2}.$$

But

$$\rho^2 = x^2 + y^2, \ \cos\varphi = \frac{x}{\rho}, \ \sin\varphi = \frac{y}{\rho}.$$

Therefore

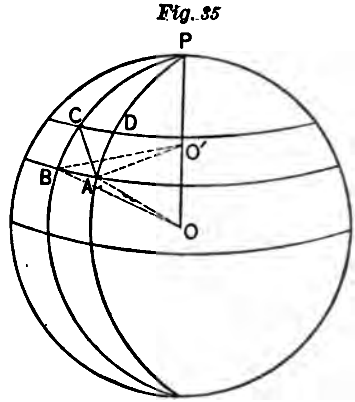
$$r = x \cos\left(\frac{1}{r}\sqrt{x^2+y^2-r^2}\right) + y \sin\left(\frac{1}{r}\sqrt{x^2+y^2-r^2}\right).$$

Such is the equation of the curve with rectangular co-ordinates. The radical $\sqrt{x^2+y^2-r^2} = TM$ represents the radius of curvature.

PROBLEM VII. A ship starting from a given latitude λ_0 and a given longitude ϑ_0, cuts at a given angle α every meridian through which it passes. In what latitude will it be when it has reached a longitude ϑ? And where will it be when it has travelled s statute miles?

Let us assume, for greater simplicity, that our globe is a perfect sphere and that the ship finds no obstacle in its way. Let $OP = r$ (Fig. 35) be the radius of the globe. When the ship reaches a place A in latitude λ and longitude ϑ, let AC be the direction of its course. Taking AC infinitesimal, we shall have

Fig. 35



$$AB - AO'.d\vartheta = r \cos\lambda . d\vartheta,$$

$$BC = BO.d\lambda = rd\lambda;$$

whence

$$\overline{AC}^2 = ds^2 = r^2 \cos^2\lambda d\vartheta^2 + r^2 d\lambda^2. \qquad (1)$$

Since $CAD = \alpha$, we have $AB = BC \tan\alpha$, that is,

$$r \cos \lambda d\vartheta = r d\lambda . \tan \alpha. \qquad (2)$$

This being squared, and substituted in (1), we have

$$ds^2 = r^2 d\lambda^2 (1 + \tan^2 \alpha) = \frac{r^2 d\lambda^2}{\cos^2 \alpha};$$

hence

$$ds = \frac{r d\lambda}{\cos \alpha}, \qquad s = \frac{r}{\cos \alpha} . \lambda + C.$$

As $s = 0$ when $\lambda = \lambda_0$, we shall have

$$C = - \frac{r}{\cos \alpha} . \lambda_0;$$

and therefore

$$s = \frac{r}{\cos \alpha} (\lambda - \lambda_0). \qquad (3)$$

From (2) we obtain $d\vartheta = \tan \alpha \dfrac{d\lambda}{\cos \lambda}$; whence (No. 77)

$$\vartheta = - \tan \alpha \log . \tan \frac{1}{2} \left(\frac{\pi}{2} - \lambda \right) + C.$$

But, when $\vartheta = \vartheta_0$, then $\lambda = \lambda_0$; hence

$$C = \vartheta_0 + \tan \alpha \log \tan \frac{1}{2} \left(\frac{\pi}{2} - \lambda_0 \right);$$

and therefore

$$\vartheta - \vartheta_0 = \tan \alpha \log \frac{\tan \frac{1}{2} \left(\frac{\pi}{2} - \lambda_0 \right)}{\tan \frac{1}{2} \left(\frac{\pi}{2} - \lambda \right)}; \qquad (4)$$

from which we obtain

$$\lambda = \frac{\pi}{2} - 2 \tan^{-1} \left[\frac{\tan \frac{1}{2} \left(\frac{\pi}{2} - \lambda_0 \right)}{e^{(\vartheta - \vartheta_0) \cot a}} \right],$$

and

$$\lambda - \lambda_0 = \left(\frac{\pi}{2} - \lambda_0 \right) - 2 \tan^{-1} \left[\frac{\tan \frac{1}{2} \left(\frac{\pi}{2} - \lambda_0 \right)}{e^{(\vartheta - \vartheta_0) \cot a}} \right].$$

This value substituted in (3) gives

$$s = \frac{r}{\cos a} \left\{ \left(\frac{\pi}{2} - \lambda_0 \right) - 2 \tan^{-1} \frac{\tan \frac{1}{2} \left(\frac{\pi}{2} - \lambda_0 \right)}{e^{(\vartheta - \vartheta_0) \cot a}} \right\} \qquad (5)$$

and the problem is solved.

As a particular case, assume

$$\lambda_0 = 0, \quad \vartheta_0 = 0, \quad a = 45°, \quad \vartheta = 2\pi.$$

Then the ship starts from the equator in the direction north-east or north-west, and traverses all the meridians. Then we have

$$\vartheta = \log \frac{1}{\tan \frac{1}{2} \left(\frac{\pi}{2} - \lambda \right)},$$

$$s = r\lambda \sqrt{2}, \quad \lambda = \frac{\pi}{2} - 2 \tan^{-1} \frac{1}{e^{2\pi}}.$$

Make $2 \tan^{-1} \frac{1}{e^{2\pi}} = z$. Then $e^{2\pi} = \cot \frac{1}{2}z$; and taking the logarithms,

$$\log . \cot \tfrac{1}{2}z = 2\pi . \log e = 6.283185 \times 0.434294 = 2.728749,$$

which corresponds to the cotangent of $0° 6' 25''.2$. Therefore

$$z = 0° 12' 50''.4, \text{ and } \lambda = 90° - 0° 12' 50''.4.$$

The ship will therefore be in latitude $89° 47' 9''.6$. Calculating the value of s and assuming the radius of the earth $= 3,960$ miles, we shall find $s = 8,774$ miles.

The assumption $\lambda = \frac{\pi}{2}$ gives $\vartheta = \infty$; hence the ship can never reach the pole. Indeed, if its course constantly makes an angle α with the meridian, it is plain that whilst the meridian traverses the pole, the ship must always be at either side of it.

Problems solved by Double or Triple Integrals.

97. A few examples will show how geometrical problems may be solved by double or triple integration.

Problem I. *A circle* AMB (Fig. 36) *revolves about the axis* OZ. *To find the surface generated by the circumference of that circle.*

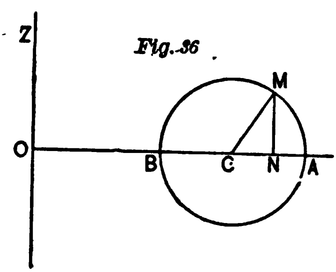


Fig. 36

Let $CM = r$, $CN = x$, $NM = y$, and $OC = R$. The distance of the point M from the axis OZ will be $R + x$; hence the infinitesimal element ds of the circumference, while describing an infinitesimal angle $d\vartheta$, will generate a surface

$$dS = (R + x)\, d\vartheta ds.$$

Therefore

$$S=\iint(R+x)\,d\vartheta ds.$$

We integrate first with respect to ϑ, which is independent both of s and of x. The integral from $\vartheta=0$ to $\vartheta=2\pi$ will be

$$S=2\pi\int(R+x)\,ds;$$

or, since

$$ds=dx\sqrt{1+\frac{dy^2}{dx^2}}=dx\sqrt{\frac{y^2+x^2}{y^2}}=\frac{rdx}{\sqrt{r^2-x^2}},$$

$$S=2\pi r\int(R+x)\,\frac{dx}{\sqrt{r^2-x^2}},$$

that is,

$$S=2\pi r\left(R\sin^{-1}\frac{x}{r}-\sqrt{r^2-x^2}\right)+C.$$

And taking the integral between $x=-r$ and $x=+r$,

$$S=2\pi r\,(R\sin^{-1}1-R\sin^{-1}-1)=2\pi r.\pi R.$$

This is the area of the surface generated by the semi-circumference AMB. The surface of the whole ring is therefore $=2\pi r.2\pi R$.

PROBLEM II. *The axes of two equal right circular cylinders intersect at right angles. To find the volume of the solid common to both.*

Let us take the origin of co-ordinates at the intersection of the axes, the one being the axis of x, the other of y; and let r be the radius of the base of the cylinders. The equations of their surfaces will be

$$x^2 + z^2 = r^2, \quad y^2 + z^2 = r^2,$$

whilst the volume of their common part will be expressed by

$$V = \iiint dxdydz.$$

Integrating first with respect to x, and reflecting that the integral must begin from $x = 0$, we have

$$V = \iint xdydz,$$

or, substituting for x its value $\sqrt{r^2 - z^2}$,

$$V = \iint dydz \sqrt{r^2 - z^2}.$$

Integrating now with respect to y from $y = 0$ to $y = \sqrt{r^2 - z^2}$, we have

$$V = \int dz \, (r^2 - z^2).$$

Integrating finally with regard to z, from $z = 0$ to $z = r$, we have

$$V = \frac{2r^3}{3}.$$

This is one-eighth of the intercepted solid. The whole is $= \frac{16r^3}{3}$.

PROBLEM III. *To find the volume of the ellipsoid with unequal axes, whose equation is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

or $$z = \frac{c}{b}\sqrt{\frac{b^2(a^2 - x^2)}{a^2} - y^2}.$$

The general formula

$$V = \iiint dx\,dy\,dz$$

being integrated from $z = 0$ to $z = z$, then substituting for z its value, we have

$$V = \frac{c}{b}\iint \left(dy\sqrt{\frac{b^2(a^2 - x^2)}{a^2} - y^2}\right)dx.$$

Integrating with respect to y, we have

$$\int dy\sqrt{\frac{b^2(a^2 - x^2)}{a^2} - y^2} = \int \frac{\frac{b^2(a^2 - x^2)}{a^2}\,dy}{\sqrt{\frac{b^2(a^2 - x^2)}{a^2} - y^2}}$$

$$-\int \frac{y^2 dy}{\sqrt{\frac{b^2(a^2 - x^2)}{a^2} - y^2}} = \frac{b^2(a^2 - x^2)}{a^2}\sin^{-1}\frac{y}{\frac{b\sqrt{a^2 - x^2}}{a}}$$

$$-\left\{-\frac{y}{2}\sqrt{\frac{b^2(a^2 - x^2)}{a^2} - y^2} + \frac{b^2(a^2 - x^2)}{2a^2}\sin^{-1}\frac{y}{\frac{b\sqrt{a^2 - x^2}}{a}}\right\} + C.$$

And, taking the integral from $y = 0$ to

$$y = \frac{b\sqrt{a^2 - x^2}}{a},$$

and reducing,

$$V = \frac{\pi}{4}\cdot\frac{bc}{a^2}\int (a^2 - x^2)\,dx.$$

This, integrated from $x=0$ to $x=a$, will give

$$V=\frac{\pi}{4}\cdot\frac{bc}{a^2}\cdot\frac{2a^3}{3}=\frac{\pi}{2}\cdot\frac{abc}{3};$$

and, as this volume is only one-eighth of the whole ellipsoid, the entire volume will be

$$\frac{4\pi abc}{3}.$$

If $a=b=c$, the ellipsoid becomes a sphere, and

$$V=\frac{4\pi a^3}{3}.$$

SECTION III.

APPLICATION OF INTEGRAL CALCULUS TO MECHANICS.

98. Before attempting the solution of any mechanical problem, it is necessary to give an exact definition of *Force*. This word *Force* is often promiscuously used as synonymous with quantity of *action*, quantity of *movement*, quantity of *pressure*, quantity of *acceleration*, etc., and a great deal of confusion has been created by this loose employment of the word.

When an agent exerts its power to produce or modify movement, *it acts;* and the quantity of its *action* is a *dynamical force*, which is measured by the change that would be produced in the quantity of movement if the action lasted for a second of time.

A quantity of *movement* is a *kinematic* or *kinetic force;* it is measured by the product of the actual velocity of the movement into the mass in motion.

A quantity of *pressure* is a *statical force;* it is measured by the quantity of movement that the pressing mass would acquire in the unit of time if all obstacle to motion were suppressed.

Though a body in movement can do *work*, as we shall explain, yet movement, as such, is not a force, but only a change of position due to an exertion of power. Velocity is the *act* or *form* of movement; hence, when a body moves with a constant velocity, its movement is said to be *uniform.* Velocity has *intensity*, but the space measured by the body has only *extension.* Time is the actuality, or duration, of movement; and if the movement be uniform, time is the ratio of its extension to its intensity, that is, the ratio of the space measured to the velocity with which it is measured.

Velocity, on the other hand, is measured and expressed by the length which it causes to be measured in the unit of time; in other words, the intensity of the movement is measured by its extension in the unit of time. Hence, if the velocity v remains unaltered, the space s measured in a number t of seconds, will be $s = vt$.

Velocity is always gradually acquired, or gradually lost, through a series of infinitesimal increments or decrements corresponding to the series of infinitesimal instants during which the body is acted upon. This is true even in the case of the so-called *instantaneous forces*, v. gr., in the communication of velocity by impact. Even in this case, the action is really continuous; and the only reason why it is called instantaneous is, that its continuation is too short to allow us to value exactly its duration. This short duration we often call *an instant*, though it is a finite length of time, and comprises a series of infinitesimal instants. It follows, that all velocity is gradually acquired, or lost, by infinitesimal degrees, through some continuous exertion of power. It also follows, that a constant continuous action, all other things being equal, is proportional to the length of time during which it is allowed to continue. If, then, an agent by its continuous action is competent to impart to a body a velocity v in the unit of time, the same agent, all other things remaining equal, will in the instant dt impart a velocity vdt; for $1 : dt :: v : vdt$.

Let us conceive a free material point, which under the continuous action of an agent A measures a space x in a time t, and at the end of this time has a velocity v. If left to itself, the point will, with this velocity, measure in the following infinitesimal instant dt an infinitesimal space dx; and as movement during an infinitesimal instant cannot but be uniform, we must have

$$dx = vdt, \text{ and } v = \frac{dx}{dt}.$$

But, if the agent A continues to act after the time t, then the velocity v in the instant dt will undergo an infinitesimal change dv. Let, then, a represent the intensity of the action at the end of the time t, Since this action will in the instant dt produce the change dv, we shall have

$$adt = dv, \text{ or } a : dv :: 1 : dt;$$

which means that the actions, all other things being equal, are proportional to their duration. From $adt = dv$, we have

$$a = \frac{dv}{dt}, \text{ or } a = \frac{d^2x}{dt^2},$$

which is the general expression of the action that modifies the velocity of the movement. Its effort $\frac{d^2x}{dt^2}$ is called *acceleration.*

If, instead of one point, we have a mass containing a number M of points, and if the agent A acts equally on each of them, its total action will be

$$Ma = M\frac{dv}{dt} = M\frac{d^2x}{dt^2};$$

and this is called a *dynamical force*, or better, the *quantity of the action* after the time t. It is *the product of the acceleration into the mass* of the body acted on.

The quantity of movement of the mass M at the end of the time t, is evidently

$$Mv, \text{ or } M\frac{dx}{dt};$$

and this is a *kinetic force*. It is *the product of the mass into its velocity.*

The quantity of pressure, or the *statical force*, is expressed by Mv like the kinetic force, v representing the velocity which the mass M would acquire in a second of time, if no obstacle existed. It is *the product of the mass into its virtual velocity.*

Work.

99. A body moving under a continuous resistance is said *to do work*. The work is by so much the greater according as a greater *mass* measures a greater *space* under a greater *resistance*. Hence the unit of work will be the work done by the unit of mass, measuring the unit of space, under a unit of resistance; and, if m be any mass, R the resistance, S the space measured, W the work, we shall have

$$W = m.R.S.$$

We are going to show that, if u be the initial velocity of the mass m, the total work which this mass can do, is always $= \frac{1}{2}mu^2$.

When the moving mass, after a time t, has measured a space x, its velocity may be expressed by $\frac{dx}{dt}$; and the resistance, which is an action modifying that velocity, and opposed to it, will be expressed by $-\frac{d^2x}{dt^2}$. Hence the infinitesimal work dW corresponding to the instant dt, will be the product of the three factors, m, dx, and $-\frac{d^2x}{dt^2}$. Therefore

$$dW = -mdx\frac{d^2x}{dt^2}.$$

Integrating this, we have

$$W = -m\int\frac{dx}{dt}\cdot\frac{d^2x}{dt} = -\tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 + C.$$

But, when $t = 0$, we have $W = 0$, and $\frac{dx}{dt} = u$. Therefore $C = \frac{1}{2}mu^2$. Hence

$$W = \tfrac{1}{2}mu^2 - \tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2.$$

When the total work has been done, then $\frac{dx}{dt} = 0$. Therefore the total work is

$$W = \tfrac{1}{2}mu^2.$$

The total work of which a mass is capable is often designated by the name of *energy*, or *work stored up* in the mass m.

EXAMPLE I. *Work under constant resistance.* A body m is thrown up vertically with an initial velocity u. The resistance is here the action of gravity, which we represent by the letter g. Hence we have

$$R = -\frac{d^2x}{dt^2} = g.$$

Multiplying both members of this equation by dx, and integrating,

$$gx = -\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + C.$$

When $t = 0$, we have $x = 0$, and $\frac{dx}{dt} = u$. Therefore $C = \frac{1}{2}u^2$; and

$$gx = \frac{1}{2}u^2 - \frac{1}{2}\left(\frac{dx}{dt}\right)^2.$$

But, when the whole work is done, then $\frac{dx}{dt} = 0$, and $x = S$. Therefore $gS = \frac{1}{2}u^2$, or, multiplying by m,

$$m.g.S = \frac{1}{2}mu^2.$$

EXAMPLE II. *Work under a resistance varying in the inverse ratio of the squared distances.* Let a mass m with the initial velocity u recede from a centre of attraction whose action at the unit of distance we shall designate by A. Let a be the original distance of m from the centre of attraction. Then, every particle of the mass m, after a time t, will be subject to a resistance or retardation

$$R = -\frac{d^2x}{dt^2} = \frac{A}{(a+x)^2}.$$

Hence

$$\frac{dx}{dt}\frac{d^2x}{dt} = -\frac{Adx}{(a+x)^2}, \quad \frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{A}{a+x} + C.$$

When $x = 0$, we have $\frac{dx}{dt} = u$; hence $C = \frac{1}{2}u^2 - \frac{A}{a}$. Substituting, and reducing,

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}u^2 - \frac{Ax}{a(a+x)}.$$

When the whole work has been done, then $\frac{dx}{dt} = 0$, and $x = S$. Hence

$$\frac{1}{2}u^2 = \frac{AS}{a(a+S)};$$

and multiplying by m,

$$m \cdot \frac{A}{a(a+S)} \cdot S = \frac{1}{2}mu^2;$$

where the resistance $\frac{A}{a(a+S)}$ is a geometrical mean between the initial resistance $\frac{A}{a^2}$ and the final resistance $\frac{A}{(a+S)^2}$.

EXAMPLE III. *Resistance proportional to the velocity of the movement.* In this case, if a be a constant, we shall have

$$\frac{d^2x}{dt^2} = -a\frac{dx}{dt}, \quad \frac{d^2x}{dt} = -a dx,$$

$$\frac{dx}{dt} = -ax + C.$$

When $x=0$, then $\frac{dx}{dt}=u$; hence $C=u$; and therefore

$$\frac{dx}{dt} = u - ax,$$

whence

$$dt = \frac{dx}{u-ax}, \quad t = -\frac{1}{a}\log(u-ax) + C'.$$

As $t=0$ gives $x=0$, hence $C' = \frac{1}{a}\log u$. Therefore

$$t = \frac{1}{a}\log\frac{u}{u-ax}.$$

When the whole work has been done, then $\frac{dx}{dt}=0$, $x=S=\frac{u}{a}$, and $t=\infty$. It would, therefore, take infinite time to measure the finite space S, and to exhaust the velocity u.

Introducing the value $S=\frac{u}{a}$ into the general equation

$$m.S.R = \tfrac{1}{2}mu^2,$$

we find $R=\frac{1}{2}au$. Thus the mean resistance is here an arithmetical mean between the initial resistance au and the final resistance *zero*. Therefore the mass m would, under a *constant* resistance $\frac{1}{2}au$ measure the space $S=\frac{u}{a}$.

EXAMPLE IV. *Resistance proportional to the space measured.* In this case, if a be a constant, we have

$$\frac{d^2x}{dt^2} = -ax, \quad \frac{dx}{dt}\frac{d^2x}{dt} = -axdx,$$

$$\left(\frac{dx}{dt}\right)^2 = -ax^2 + C.$$

When $x = 0$, then $\frac{dx}{dt} = u$. Hence $C = u^2$. Therefore

$$\left(\frac{dx}{dt}\right)^2 = u^2 - ax^2.$$

When the whole work has been done, then $\frac{dx}{dt} = 0$, $x = \frac{u}{\sqrt{a}} = S$. Hence

$$m.S.R = m\frac{u}{\sqrt{a}}R = \tfrac{1}{2}mu^2;$$

and therefore $R = \frac{1}{2}u\sqrt{a}$. This mean resistance is an arithmetical mean between the initial resistance *zero* and the final resistance $u\sqrt{a}$.

It may be observed that the integration of $\left(\frac{dx}{dt}\right)^2 = u^2 - ax^2$ gives

$$t = \frac{1}{\sqrt{a}}\sin^{-1}\frac{x}{u}\sqrt{a},$$

which, when $x = S = \frac{u}{\sqrt{a}}$, reduces to

$$t=\frac{\sin^{-1}1}{\sqrt{a}}=\frac{\pi}{2\sqrt{a}}.$$

This shows that the time employed in doing the whole work would be independent of the initial velocity, and of the space measured.

Movement Uniformly Varied.

100. When the velocity of a mass m increases or decreases uniformly, its movement is uniformly accelerated or retarded; which implies that the action, under which the movement varies, must be constant. Assume $m=1$, and let the accelerating action be g. Then the equation of the movement will be $\frac{d^2s}{dt^2}=g$; whence

$$\frac{ds}{dt}=gt+C, \text{ and } s=\tfrac{1}{2}gt^2+Ct+C',$$

s being the space measured at the end of the time t. The constant C is what the velocity $\frac{ds}{dt}$ becomes when $t=0$; that is, C is the *initial velocity*. Hence, if there be no initial velocity, C will be $=0$. The constant C' is what s becomes when $t=0$; that is, C' is the *initial space*, or a space already measured before the beginning of the time t. Hence, when the space is reckoned from the beginning of the time t, C' will be $=0$. Supposing therefore $C=0$ and $C'=0$ in the above equations, the velocity v acquired in the time t will be

$$v=gt \qquad (1)$$

and the space traversed in the same time

$$s = \tfrac{1}{2}gt^2. \qquad (2)$$

This last equation, being multiplied by $2g$, gives $2gs = g^2t^2$, or

$$v^2 = 2gs, \text{ and } v = \sqrt{2gs}, \qquad (3)$$

that is, the velocity acquired or lost by a body while measuring a space s with uniformly accelerated or retarded movement, is equal to the square root of twice the product of such space into the accelerating or retarding action. Such a velocity is styled the velocity *due to the space* s, whilst s itself is called the space *due to the velocity* v.

Movement not Uniformly Varied.

101. The attraction of the earth, and of planets in general, varies inversely as the squared distances of the bodies attracted. Let r be the radius of the earth, g the intensity of its attraction at its surface, g' the intensity of its attraction at a distance s. Then

$$g' : g :: r^2 : s^2, \text{ whence } g' = g\frac{r^2}{s^2}.$$

We shall have, therefore,

$$\frac{d^2s}{dt^2} = -g\frac{r^2}{s^2},$$

where the second member has the negative sign, because the action tends to diminish the distance s.

Multiplying this by $2ds$, and integrating, we have

$$\left(\frac{ds}{dt}\right)^2 = 2g\,\frac{r^2}{s} + C.$$

Let $s = h$ when $t = 0$; then $\frac{ds}{dt} = 0$, and

$$C = -2g\,\frac{r^2}{h};$$

hence

$$\left(\frac{ds}{dt}\right)^2 = v^2 = 2gr^2\left(\frac{1}{s} - \frac{1}{h}\right). \qquad (1)$$

Such is the velocity acquired by the body while falling from the original height h to the height s.

Let $h = \infty$, and $s = r$. Then equation (1) becomes

$$v^2 = 2gr, \text{ hence } v = \sqrt{2gr},$$

and, as $g = 32.088$ feet, $r = 20{,}923{,}596$ feet, we shall find

$$v = 36{,}644 \text{ feet.}$$

Thus a body falling upon the earth from an infinite distance would reach its surface with a velocity of nearly 7 miles a second.

The attraction of the sun at its surface is $g = 890.16$ feet, and its radius is $r = 430{,}854.5$ miles. With these data, we find that a body falling upon the sun from an infinite distance would have a final velocity of about 381 miles per second.

From equation (1), extracting the root, and taking the negative signs (as ds diminishes when dt

increases), we obtain

$$dt = -\sqrt{\frac{h}{2gr^2}} \cdot \frac{ds\sqrt{s}}{\sqrt{h-s}} = -\sqrt{\frac{h}{2gr^2}} \cdot \frac{sds}{\sqrt{hs-s^2}};$$

which may be written thus,*

$$dt = \sqrt{\frac{h}{2gr^2}} \cdot \frac{\frac{h}{2} - s - \frac{h}{2}}{\sqrt{hs-s^2}}\, ds =$$

$$\sqrt{\frac{h}{2gr^2}} \left\{ \frac{1}{2} \cdot \frac{(h-2s)\,ds}{\sqrt{hs-s^2}} - \frac{\frac{h}{2}\, ds}{\sqrt{hs-s^2}} \right\}.$$

Hence

$$t = \sqrt{\frac{h}{2gr^2}} \left\{ \sqrt{hs-s^2} - \frac{h}{2} \operatorname{vers}^{-1} \frac{2s}{h} \right\} + C.$$

But, when $t = 0$, we have $s = h$. Therefore

$$C = \frac{h}{2}\pi \sqrt{\frac{h}{2gr^2}}.$$

And therefore

$$t = \sqrt{\frac{h}{2gr^2}} \left\{ \sqrt{hs-s^2} + \frac{h}{2}\left(\pi - \operatorname{vers}^{-1} \frac{2s}{h}\right) \right\}.$$

This equation, when $s = r$, will give the time required for a body to fall from a distance h to the surface of the attracting body.

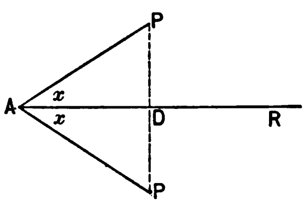
The mean distance of the moon from the earth is $60r$, r being the radius of the earth. The time required for a body to fall from the moon to the earth would be $t = 417{,}381$ seconds, that is to say, 4 days, 19 hours, 56 minutes, and 21 seconds.

Composition and Decomposition of Forces.

102. Before we proceed further, we must say a few words on the composition and decomposition of forces. *Dynamical forces* are usually represented by lines proportional to the velocities which they impart in the unit of time; *kinetic forces* by lines proportional to the spaces measured in the unit of time; *statical forces* by lines proportional to the spaces that would be measured in the unit of time, if no obstacle existed.

Let two equal forces P and P (Fig. 37) be applied to a point A, and let them make with each other an angle $PAP=2x$. If we can find a single force R, which applied to A will produce the same effect as the two forces P, this new force will be called the *resultant* of the forces P, whilst the forces P will be its *components*. The resultant will evidently lie in the plane of its components; and when the two components are equal, it will bisect their angle.

Fig. 37



The value of the resultant depends both on the intensity of its components, and on the angle at which they meet. Hence the resultant R of the two equal forces P may be expressed by

$$R=2Pf(x),$$

$f(x)$ being a trigonometric function of the angle x. This function is easily determined. For we know, that when $x=0°$, the resultant is $R=2P$. Hence $f(0°)=1$. We know, also, that when $x=60°$, the resultant is $R = P$. Hence $f(60°) = \frac{1}{2}$. We know in like manner, that when $x = 90°$, the resultant is $R = 0$, as the two forces then neutralize each other. Hence $f(90°) = 0$. Now, the only trigonometric function which can satisfy these conditions is the cosine of x; for

$$\cos 0° = 1, \ \cos 60° = \tfrac{1}{2}, \ \cos 90° = 0.$$

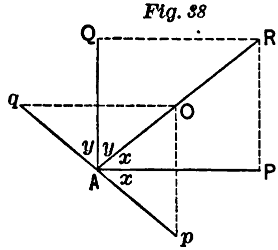
Therefore the resultant of the two equal forces P is

$$R = 2P \cos x.$$

Drawing PP, which will intersect AR at right angles in D, we have $AD = P \cos x$; and therefore $R = 2AD = AR$. Thus *the resultant is represented as to its intensity and direction by the diagonal of the rhombus constructed on the two equal components.*

103. Let now two unequal forces P and Q be applied to the point A at right angles (Fig. 38), and let x and y be the angles which the resultant shall make with P and Q respectively.

Fig. 38



If P be conceived as a resultant of two equal forces p that make with it an angle x, one of these components will lie in the direction of the resultant, and the other in the direction Ap. If, in like manner, Q be conceived as a resultant of two equal forces q that make with it an angle y, one of these components will lie in the direction of the resultant, and the other in the direction Aq. And we shall have (No. 102)

$$P = 2p \cos x, \quad Q = 2q \cos y.$$

Now, evidently, the resultant is the sum of the two forces p and q which lie in its direction, while the other two forces p and q which lie in the direction Ap and Aq, are directly opposite, and must neutralize each other. Therefore $p = q$, and $R = 2p = 2q$. Hence

$$R = \frac{P}{\cos x} = \frac{Q}{\cos y};$$

or $$R \cos x = P, \; R \cos y = Q.$$

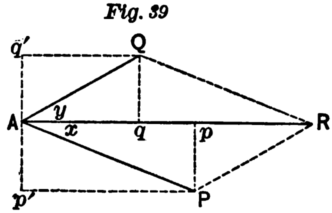
Squaring these equations, adding them together and remarking that in our case $\cos y = \sin x$, we obtain

$$R^2 = P^2 + Q^2,$$

that is, *the resultant of two forces meeting at a right angle is the diagonal of the rectangle constructed on its components.*

104. If the forces P and Q do not form a right angle (Fig. 39), then considering P as the diagonal of a rectangle of which one side Ap lies on the resultant, and Q as the diagonal of a rectangle of which one side Aq lies on the resultant, then the resultant will be the sum of Ap and Aq, and the other components Ap' and Aq' will be opposite and destroy each other. Now, if x and y

Fig. 39



be the angles that P and Q make with the resultant, we shall have

$$R = Ap + Aq = P \cos x + Q \cos y.$$

Squaring, and changing $\cos^2 x$ and $\cos^2 y$ into $1 - \sin^2 x$, and $1 - \sin^2 y$, we have

$$R^2 = P^2 + Q^2 - P^2 \sin^2 x - Q^2 \sin^2 y + 2PQ \cos x \cos y.$$

But, as $Ap' = Aq'$, we have $Pp = Qq$, or

$$P \sin x = Q \sin y;$$

hence

$$P^2 \sin^2 x = PQ \sin x \sin y = Q^2 \sin^2 y$$

and

$$P^2 \sin^2 x + Q^2 \sin^2 y = 2PQ \sin x \sin y.$$

Therefore, substituting,

$$R^2 = P^2 + Q^2 + 2PQ (\cos x \cos y - \sin x \sin y),$$

or finally

$$R^2 = P^2 + Q^2 + 2PQ \cos (x + y);$$

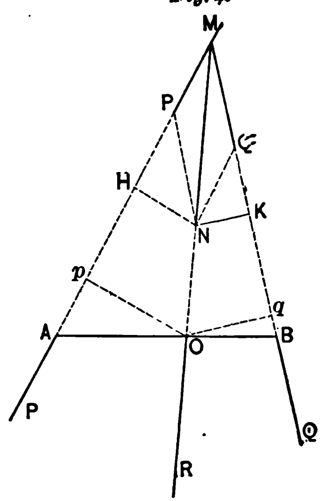
which is the expression for the diagonal of a parallelogram whose sides P and Q contain the angle $x + y$. *Hence the resultant of two forces meeting at any angle is the diagonal of the parallelogram constructed on those forces as sides.*

When the forces are more than two, their resultant is obtained by successive compositions. On the other hand, every resultant can be resolved into components, in many manners; for the same line may be the diagonal of many different parallelograms.

Moments.

105. Let a rigid rod AB (Fig. 40) be acted on by a force P applied at A, and by a force Q applied at B. If the directions of PA and QB converge towards a point M, we may, without altering the system, apply the two forces at M, and find their resultant MN. Prolonging MN towards R, and taking $OR = MN$, the line OR will represent the resultant of P and Q applied to the rod at the point O, and producing the same effect as the forces P and Q applied at A and B.

Fig. 40



To determine the position of the point O on the rod, draw NH and Op perpendicular to AM, also NK and Oq perpendicular to BM. The similar triangles NPH and NQK will give

$$NH : NK :: NP : NQ :: Q : P.$$

On the other hand

$$NH : NK :: Op : Oq;$$

therefore

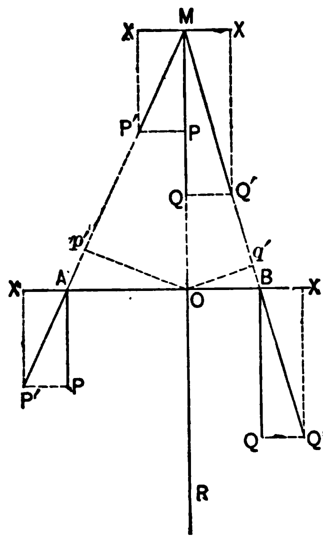
$$P : Q :: Oq : Op;$$

or making $Op = p$, and $Oq = q$,

$$P : Q :: q : p, \text{ whence } Pp = Qq.$$

If the forces P and Q were parallel, we could, without altering the system, apply at the extremities A and B of the rod (Fig. 41) the equal and opposite forces AX and BX, which being compounded with P and Q respectively, would give the two converging resultants AP' and BQ'. These resultants would now be transferred along their direction to M, and decomposed there, the first into P and MX, the second into Q and MX; and since the forces MX destroy each other, there would remain only P and Q; that is, the resultant would be $P + Q$. Prolonging MP towards R, and taking $OR = P + Q$, the force OR applied to the rod at the point O will produce the same effect as the two forces P and Q applied at A and B.

Fig. 41



To determine the position of the point O on the rod, draw Op' perpendicular to AM, and Oq' perpendicular to BM. Then, by the preceding demonstration, $P' : Q' :: Oq' : Op'$; and

$$P' . Op' = Q' . Oq'.$$

But the similar triangles AOp' and PAP' give

$$P : P' :: Op' : OA,$$

and the similar triangles BOq' and QBQ' give

$$Q : Q' :: Oq' : OB;$$

therefore

$$P.OA = P'.Op', \text{ and } Q.OB = Q'.Oq'.$$

Hence we have also

$$P.OA = Q.OB,$$

or, making

$$OA = p, \ OB = q, \ Pp = Qq,$$

as in the preceding case.

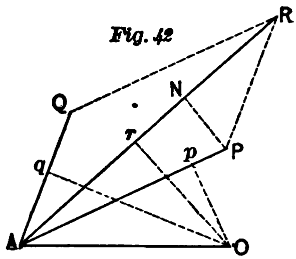
The products Pp and Qq are called *moments.* Hence the moment of a force with respect to a point O is the product of the force into the perpendicular drawn from O to the line of its action. The moment is the expression of the effort with which a force tends to turn the rod about the point O, when this point is fixed. In this case, the fixity of the point O will neutralize the resultant of the actions; and the equality of the moments Pp and Qq shows that the efforts of the forces P and Q balance each other, and that the rod is in equilibrium.

When the forces P and Q are parallel, their resultant is parallel to them, as is evident from the mode of its construction; and the same is true of the resultant of any number of parallel forces.

106. When two forces, or more, act in the same plane on the same point, *the moment of the resultant taken with respect to a fixed point is equal to the algebraic sum of the moments of its components with respect to the same fixed point.*

Let P and Q (Fig. 42) be two forces acting on the point A, and let R be their resultant. Let AO be a rigid rod which can revolve about the fixed point O. This point O will then be styled *the centre of moments*. Draw Op, Oq, Or perpendicular to AP, AQ, AR respectively. Denote the angle RAP by α, RAQ (or its equal ARP) by β, and RAO by φ. Since $PR = Q$, we have from the triangles APN and NPR the equations

Fig. 42



$$R = Q \cos \beta + P \cos \alpha,$$

$$0 = Q \sin \beta - P \sin \alpha.$$

Multiplying the first by $\sin \varphi$ and the second by $\cos \varphi$, and adding together the results, we find

$$R \sin \varphi = Q (\sin \varphi \cos \beta + \sin \beta \cos \varphi) \\ + P (\sin \varphi \cos \alpha - \sin \alpha \cos \varphi),$$

or

$$R \sin \varphi = Q \sin (\varphi + \beta) + P \sin (\varphi - \alpha).$$

But

$$\sin \varphi = \frac{Or}{AO}, \quad \sin (\varphi + \beta) = \frac{Oq}{Ao},$$

$$\sin (\varphi - \alpha) = \frac{Op}{AO};$$

substituting in the preceding equation, making

$$Or = r, \ Op = p, \ Oq = q,$$

and reducing, we have

$$Rr = Pp + Qq,$$

as was to be proved. And, if we regard Q as a resultant of two forces, and these again as the resultants of other forces, our conclusion can be extended to any number of concurrent forces acting in the same plane, and thus we may write in general

$$Rr = Pp + P'p' + P''p'' + P'''p''' + \ldots$$

When the point O lies within the angle formed by the resultant and any of its components, then the moments of such components are negative; for $\sin(\varphi - \alpha)$ is then negative. Hence the above sum of moments is algebraical, not arithmetical.

Virtual Moments.

107. When two forces P and Q applied at the extremities A and B of the lever AB (Fig. 43) are in equilibrium about the fixed point O, if the equilibrium be disturbed or endangered by an extraneous force, the points A and B will describe, or tend to describe, similar arcs AA' and BB' about the point O; and we shall have the proportion

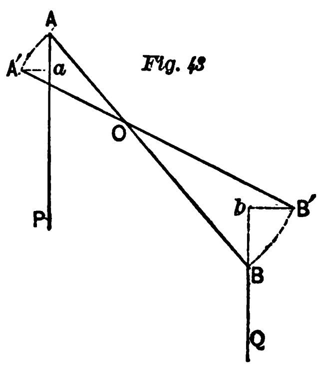


Fig. 43

$$AA' : BB' :: OA : OB.$$

Projecting AA' on the direction of P, and BB' on the direction of Q, we shall have also

$$Aa : Bb :: AA' : BB';$$

and therefore

$$Aa : Bb :: OA : OB;$$

and because $OA : OB :: Q : P$, therefore

$$Aa : Bb :: Q : P, \text{ or } P.Aa = Q.Bb.$$

The lines Aa and Bb represent the so-called *virtual velocities* of the forces P and Q. Not that these forces have any velocity at all, but because Aa and Bb are the measure of the virtual velocities which rule the ascent or descent of the points of application of the forces. The former, Aa, which falls on AP is considered *positive;* the latter Bb which falls on the prolongation of BQ is taken as *negative.* The products $P.Aa$, and $Q.Bb$ are called the *virtual moments* of P and Q.

The virtual moments of P, Q, and their resultant R (Fig. 42) are $P.Ap$, $Q.Aq$, $R.Ar$. Resuming the equations

$$R = Q \cos \beta + P \cos \alpha,$$

$$0 = Q \sin \beta - P \sin \alpha,$$

multiplying the first by $\cos \varphi$, the second by $\sin \varphi$, and subtracting the second from the first, we find

$$R \cos \varphi = P (\cos \alpha \cos \varphi + \sin \alpha \sin \varphi) + Q (\cos \beta \cos \varphi - \sin \beta \sin \varphi);$$

or

$$R \cos \varphi = P \cos (\varphi - \alpha) + Q \cos (\varphi + \beta).$$

But

$$\cos \varphi = \frac{Ar}{Ao}, \quad \cos (\varphi - \alpha) = \frac{Ap}{Ao},$$

$$\cos(\varphi + \beta) = \frac{Aq}{Ao}.$$

Substituting in the preceding equation, and reducing, we have

$$R.Ar = P.Ap + Q.Aq;$$

that is, *the virtual moment of the resultant is equal to the algebraic sum of the virtual moments of its components.* This conclusion extends to the resultant of any number of forces that lie in the same plane.

The virtual velocities Ar, Ap, Aq, etc., are usually represented by δr, δp, δq, etc. Hence we may write, in general,

$$R\delta r = P\delta p + P'\delta p' + P''\delta p'' + \ldots$$

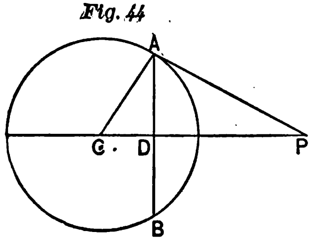
where the character δ denotes an infinitesimal variation corresponding to an infinitesimal instant dt.

The point O with regard to which the virtual moments are taken, is *the centre of the virtual moments.* When the algebraic sum of the virtual moments of the forces is $=0$, the system evidently is in equilibrium about the point O.

Attraction of a Sphere on a Material Point.

108. Let C (Fig. 44) be the centre of a spherical shell of infinitesimal thickness, and P a material point at some distance from the shell. If the mass of the shell be denoted by m, the mass of the infinitesimal zone AB, whose centre lies at a distance $CD = x$ from the centre of the shell, will

Fig. 44



be expressed by $dm = \frac{mdx}{2r}$, r being the radius of the shell. The action of this infinitesimal mass upon the point P is the resultant of the equal actions of all its particles, and is equal to the sum of such actions multiplied by the cosine of the angle APD (No. 102).

Let $AP = z$. By the Newtonian law, if the action of a particle at the unit of distance be $= 1$, its action at the distance z will be $= \frac{1}{z^2}$. Hence the action $d\varphi$ of the zone AB on the point P will be

$$d\varphi = \frac{mdx}{2r} \cdot \frac{1}{z^2} \cos APD.$$

But

$$\cos APD = \frac{PD}{AP} = \frac{CP - CD}{z} = \frac{a - x}{z},$$

where $a = CP$. On the other hand in the triangle PAC we have

$$\overline{AP}^2 = z^2 = a^2 + r^2 - 2ar \cos ACD = a^2 + r^2 - 2ax;$$

therefore

$$d\varphi = \frac{m}{2r} \frac{(a - x)\,dx}{\sqrt{(a^2 + r^2 - 2ax)^3}},$$

or

$$d\varphi = \frac{m}{2r} \left\{ \frac{adx}{\sqrt{(a^2 + r^2 - 2ax)^3}} - \frac{xdx}{\sqrt{(a^2 + r^2 - 2ax)^3}} \right\}.$$

Now

$$\int \frac{adx}{\sqrt{(a^2 + r^2 - 2ax)^3}} = \frac{1}{\sqrt{a^2 + r^2 - 2ax}},$$

and

$$\int \frac{xdx}{\sqrt{(a^2+r^2-2ax)^3}} =$$

$$\frac{x}{a\sqrt{a^2+r^2-2ax}} + \frac{\sqrt{a^2+r^2-2ax}}{a^4}.$$

Hence, substituting, and reducing,

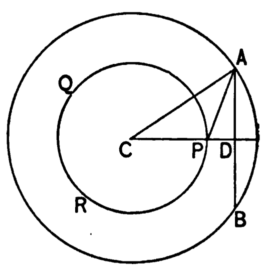
$$\varphi = \frac{m}{2r} \cdot \frac{ax-r^2}{a^2\sqrt{a^2+r^2-2ax}} + C;$$

and taking the integral from $x=-r$ to $x=r$, and reducing,

$$\varphi = \frac{m}{2ra^2}\left\{\frac{r(a-r)}{a-r} + \frac{r(a+r)}{a+r}\right\} = \frac{m}{a^2};$$

that is, the total action is the same as if the whole mass of the shell were concentrated in *C*. The same being true of all the shells into which a solid sphere may be decomposed, it follows that the action of a sphere formed of homogeneous shells is the same as if its mass were concentrated in its centre.

Fig. 45



109. If the point *P* were placed anywhere within the spherical shell (Fig. 45), the resultant of all the actions of the shell upon it would constantly be $=0$, and therefore the point *P* would remain in equilibrium. For, in this case, the differential equation would be

$$d\varphi = \frac{m}{2r} \cdot \frac{(x-a)\,dx}{\sqrt{(a^2+r^2-2ax)^3}},$$

which differs only by the sign from the equation of the former case. Hence the integral will be

$$\varphi = \frac{m}{2r} \cdot \frac{r^2 - ax}{a^2 \sqrt{a^2 + r^2 - 2ax}} + C;$$

and this, if taken from $x = -r$ to $x = r$, will give

$$\varphi = \frac{m}{2ra^2} \left\{ \frac{r(r-a)}{r-a} - \frac{r(r+a)}{r+a} \right\} = 0.$$

And this is true of all the shells whose radius is greater than CP. Accordingly, if the point P were placed within a solid sphere, it would be attracted as if the shells beyond CP had no existence; that is, it would only feel the attraction of the nucleus PQR.

Corollary. If an opening were made along one of the diameters of the earth, and a body allowed to fall through it, the body (abstraction being made from the resistance of the air) would be urged towards the centre by an action varying as the simple distance from the centre.

For, let r be the radius, and ρ the density (supposed uniform) of the earth. Its mass will then be $\frac{4\pi r^3}{3}\rho$; whilst the mass of the nucleus PQR will be $\frac{4\pi s^3}{3}\rho$, s being its radius. Now, the action of the earth at its surface is $g = \frac{4\pi r^3}{3}\rho\frac{\varphi}{r^2}$, and the action of the nucleus at a distance s from the centre is $g' = \frac{4\pi s^3}{3}\rho\frac{\varphi}{s^2}$; whence

$$g' : g :: s : r, \text{ or } g' = g\frac{s}{r}.$$

Hence the equation for the movement of a point approaching the centre, is

$$\frac{d^2s}{dt^2} = -\frac{gs}{r}.$$

Multiplying by $2ds$, and integrating, we have

$$\left(\frac{ds}{dt}\right)^2 = -\frac{gs^2}{r} + C.$$

Making $\frac{ds}{dt} = 0$ when $s = r$, we have $C = \frac{gr^2}{r}$. Therefore

$$\left(\frac{ds}{dt}\right)^2 = v^2 = \frac{g}{r}(r^2 - s^2). \qquad (1)$$

When the body reaches the centre, then $s = 0$, and $v = \sqrt{gr}$, which is the maximum velocity. When the body has reached the centre, its velocity will carry it further on, and, as s changes its sign, the motion will be retarded instead of accelerated, until v reduces to zero when $s = -r$. Then the body will fall again towards the centre, and measure backward the same diameter, and perform a continuous series of oscillations of the same kind.

Extracting the root of equation (1) and taking the radical negatively, because ds and dt have opposite signs, we find

$$dt = -\sqrt{\frac{r}{g}} \cdot \frac{ds}{\sqrt{r^2 - s^2}};$$

and this integrated from $s = r$ to $s = -r$, gives

$$t = \pi\sqrt{\frac{r}{g}}.$$

This is the time of one entire excursion. This time is equal to that in which a body would measure the semi-circumference πr with a uniform velocity $= \sqrt{gr}$; for, if $t\sqrt{gr} = \pi r$, then $t = \pi\sqrt{\frac{r}{g}}$. .

The time t of the excursion is independent of the distance from which the body begins to fall. For, since we have

$$r : g :: s : g',$$

we can replace the radical $\sqrt{\frac{r}{g}}$ by $\sqrt{\frac{s}{g'}}$, without altering the value of t. Hence all the excursions will be isochronous, whatever may be their amplitude. But these results would be greatly modified by the resistance of the air, which we have neglected.

Centre of Gravity.

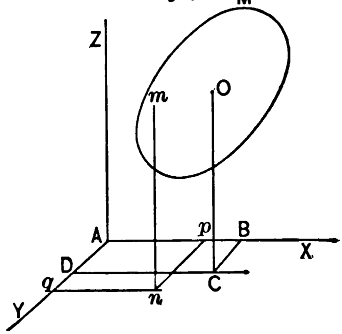
110. The centre of gravity of a body is a point within the body, through which the resultant of the actions of gravity on each particle of the body always passes. All these actions are directed towards the centre of the earth; yet they may, without error, be considered parallel. Hence their resultant is their sum (No. 105).

It is obvious that the centre of gravity of a straight line is at its middle point; also, that the centre of gravity of a plane figure is in that plane, and if the figure has a line of symmetry, its centre of gravity is on that line. In like manner, if a solid has a plane of symmetry, its centre of gravity is in that plane.

The centre of gravity of a *homogeneous* body does not depend on the intensity of gravity or on the density of the body. Its position depends only on the form of its volume. We may therefore substitute volumes for masses and weights, and consider only the relative position of the elements of which the body is composed.

Let M (Fig. 46) be a homogeneous body of any form. Draw rectangular axes, and let the plane XY be horizontal. The action of gravity will be parallel to the axis AZ. Let m be an element of the body, and let its co-ordinates be $x = qn$, $y = pn$, $z = mn$. If dv be the volume of the element m, its moment with respect to the axis AY will be xdv. Every other element of the body M will give a similar moment, the value of x varying between the limits of the body. Hence the sum of the moments of all the elements with respect to the axis AY will be $\int xdv$.

Fig. 46



Let now O be the centre of gravity of the body, and $x_0 = DC$, $y_0 = BC$, $z_0 = OC$, its co-ordinates. Since the resultant of the actions of gravity passes through O, the moment of the resultant with respect to the axis AY will be $x_0 \int dv$. Hence, by the theory of moments (No. 106),

$$x_0 \int dv = \int xdv ;$$

and therefore

$$x_0 = \frac{\int x dv}{\int dv}.$$

If the moments were taken with respect to the axis AX, we would find in like manner

$$y_0 = \frac{\int y dv}{\int dv};$$

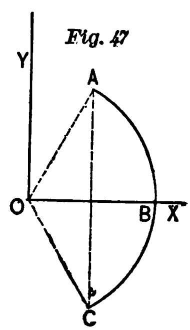
and if the figure were turned about so as to make the axis of x vertical, we would have also

$$z_0 = \frac{\int z dv}{\int dv}.$$

Such are the values of the co-ordinates of the centre of gravity of the body.

111. *Centre of gravity of a circular arc.* Let the axis OX (Fig. 47) bisect the arc ABC. Then OX will be a line of symmetry, and the centre of gravity will lie on OX. Let $AC = c$ be the chord, and $OA = r$ the radius of the circular arc. If O be the origin of co-ordinates, the equation of the circle will be

Fig. 47



$$y^2 = r^2 - x^2;$$

and therefore

$$dv = \sqrt{dx^2 + dy^2} = dy \sqrt{\frac{y^2 + x^2}{x^2}} = r\,\frac{dy}{x} = \frac{r dy}{\sqrt{r^2 - y^2}}.$$

Substituting in the above expression for x_0, we have

$$x_0 = \frac{\int r dy}{\int \frac{r dy}{\sqrt{r^2 - y^2}}},$$

and integrating from $y = -\frac{1}{2}c$ to $y = \frac{1}{2}c$,

$$x_0 = \frac{rc}{r\left(\sin^{-1}\frac{c}{2r} - \sin^{-1}\left(-\frac{c}{2r}\right)\right)} = \frac{rc}{\text{arc}.ABC}.$$

Making the angle $AOX = \vartheta$, we have $c = 2r \sin \vartheta$, whilst the arc $ABC = 2r\vartheta$; whence we get also

$$x_0 = r\,\frac{\sin \vartheta}{\vartheta}.$$

112. *Centre of gravity of a circular segment.* Referring to Fig. 47, where the segment ABC is bisected by the axis OX, we find

$$dv = 2ydx = \frac{2y^2 dy}{\sqrt{r^2 - y^2}}, \quad xdv = 2y^2 dy;$$

whence

$$x_0 = \frac{\int 2y^2 dy}{\int \frac{2y^2 dy}{\sqrt{r^2 - y^2}}};$$

and integrating from $y = 0$ to $y = y$, by formula (33),

$$x_0 = \frac{\frac{2}{3}y^3}{r^2 \sin^{-1}\frac{y}{r} - y\sqrt{r^2 - y^2}},$$

and, since $y = r \sin \vartheta$, substituting and reducing, we have

$$x_0 = \frac{2r}{3} \cdot \frac{\sin^3 \vartheta}{\vartheta - \sin \vartheta \cos \vartheta}.$$

113. *Centre of gravity of a circular sector.* Referring again to Fig. 47, let us conceive that the sector $OABC$ has been divided into equal infinitesimal sectors, every one of which may be looked upon as a triangle having the vertex in O and an infinitesimal base on the circumference. The centre of gravity of every one of these sectors, or triangles, will be at a distance $\frac{2}{3}r$ from the centre O, as can be shown by a simple geometric construction. Hence the whole series of these centres of gravity will determine an arc of circle similar to the arc ABC, but having a radius $\frac{2}{3}r$. As the matter of each elementary sector can be considered concentrated in its centre of gravity, it follows that the centre of gravity of the whole sector is the same as the centre of gravity of said arc. Hence, applying to our case the result of No. 111,

$$x_0 = \frac{\frac{2}{3}r \times \frac{2}{3}c}{\frac{2}{3} \text{ arc } ABC} = \frac{2r}{3} \cdot \frac{c}{\text{arc } ABC} = \frac{2r}{3} \cdot \frac{\sin \vartheta}{\vartheta}.$$

114. *Centre of gravity of a parabolic area.* In the parabola, whose equation is $y^2 = 2px$, we have for an area comprised between the curve and a double ordinate,

$$dv = 2y\,dx = 2\sqrt{2p}\,.\,x^{\frac{1}{2}}dx, \qquad x\,dv = 2\sqrt{2p}\,.\,x^{\frac{3}{2}}dx;$$

hence

$$x_0 = \frac{\int x^{\frac{3}{2}} dx}{\int x^{\frac{1}{2}} dx}.$$

Let a be the terminal abscissa of the area in question. Integrating between the limits $x = 0$ and $x = a$, we shall have

$$x_0 = \tfrac{3}{5} a,$$

a value independent of the parameter of the curve.

115. *Centre of gravity of a paraboloid of revolution.* In this case

$$dv = \pi y^2 dx, \quad xdv = \pi y^2 x dx,$$

that is,

$$dv = 2\pi . pxdx, \quad xdv = 2\pi . px^2 dx.$$

Hence

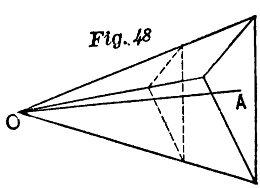
$$x_0 = \frac{\int x^2 dx}{\int x dx};$$

and integrating from $x = 0$ to $x = a$, and reducing,

$$x_0 = \tfrac{2}{3} a,$$

a value independent of the parameter of the curve.

116. *Centre of gravity of a right pyramid.* Let O (Fig. 48) be the vertex, and $OA = h$ the height of the pyramid. Calling b the base, any section parallel to it at a distance x from the vertex, will be $= b \frac{x^2}{h^2}$, and its infinitesimal volume will be $\frac{bx^2 dx}{h^2}$. Hence



$$dv = \frac{b}{h^2}\,x^2dx, \quad \text{and} \quad xdv = \frac{b}{h^2}\,x^3dx;$$

therefore

$$x_0 = \frac{\int x^3dx}{\int x^2dx};$$

and integrating from $x = 0$ to $x = h$, and reducing,

$$x_0 = \tfrac{3}{4}h.$$

For the *frustum of a pyramid*, the integral is to be taken from $x = h'$ to $x = h$. This would give

$$x_0 = \frac{3}{4} \cdot \frac{h^4 - h'^4}{h^3 - h'^3}.$$

If the base of the pyramid becomes a circle, the above equations will give the centre of gravity of the cone, and of the frustum of a cone.

117. *Centre of gravity of a spherical zone.* We have from Geometry

$$dv = 2\pi r\,.dx, \quad \text{and} \quad xdv = 2\pi r\,.xdx;$$

whence

$$x_0 = \frac{\int xdx}{\int dx};$$

or, integrating from $x = a$ to $x = r$, and reducing,

$$x_0 = \frac{1}{2} \cdot \frac{r^2 - a^2}{r - a} = \tfrac{1}{2}(r + a).$$

If $a = 0$, we have $x_0 = \frac{1}{2}r$ for the centre of gravity of the surface of a hemisphere.

118. *Centre of gravity of a spherical sector.* The spherical sector may be conceived as a sum of equal infinitesimal pyramids having a common vertex in the centre of the sphere, and an infinitesimal base on its surface. Each such pyramid has its centre of gravity at a distance $\frac{3}{4}r$ from the centre of the sphere; so that we may consider the surface passing through all such centres of gravity as forming a spherical zone with a radius $\frac{3}{4}r$. Hence the centre of gravity of the whole spherical sector will be found by substituting $\frac{3}{4}r$ and $\frac{3}{4}a$ for r and a in the result of No. 117. Therefore

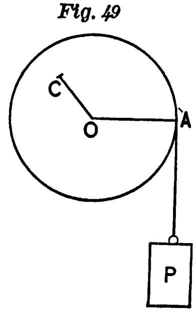
$$x_0 = \tfrac{3}{8}(r + a).$$

If $a = 0$, then $x_0 = \frac{3}{8}r$ will be the distance of the centre of gravity of a solid hemisphere.

Moment of Inertia.

119. Let $OA = r$ (Fig. 49) be the radius of a cylinder having a mass m. If its axis be horizontal, and a weight P be attached to the cylinder by a string wrapped around its surface, the cylinder will be caused to revolve about its axis. Let $\frac{d\vartheta}{dt}$ be the angular velocity imparted to every element of the mass at the time t. Then any element dm, whose distance from the axis is $x = OC$, will have a velocity $x\,\frac{d\vartheta}{dt}$,

Fig. 49



and its quantity of movement will be $dm . x\,\frac{d\vartheta}{dt}$.

The accelerating action of P on such an element will be, at this instant, $dm.x\,\frac{d^2\vartheta}{dt^2}$, and its moment with respect to the axis will be $dm.x^2\,\frac{d^2\vartheta}{dt^2}$. Hence the moment of the action of P on the whole mass m will be the sum of all such moments (No. 106), that is,

$$\frac{d^2\vartheta}{dt^2}\,(x^2dm + x'^2dm' + x''^2dm'' + \ldots),$$

or, briefly,

$$\frac{d^2\vartheta}{dt^2}\int x^2dm.$$

But the moment of the action of P is also expressed by Pr; hence

$$\frac{d^2\vartheta}{dt^2}\int x^2dm = Pr, \text{ and } \frac{d^2\vartheta}{dt^2} = \frac{Pr}{\int x^2dm}.$$

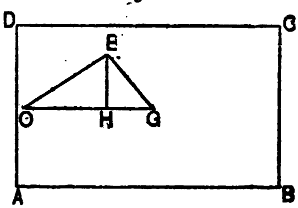
The quantity $\int x^2dm$ is called the *moment of inertia* of the mass m with respect to the axis passing through its centre of gravity. If by performing the integration we find

$$\int x^2dm = mk^2, \tag{1}$$

k is called the *radius of gyration*, inasmuch as the movement will be the same as if the whole mass m were collected together at the distance k from the axis of rotation.

120. It is to be remarked that the moment of inertia varies in the same body according to the position of the axis of rotation. To investigate the law of its variation, let $ABCD$ (Fig. 50) be a section of the mass m by a plane perpendicular to the axis of rotation, O the point where the axis is cut by this plane, and G the point where a parallel axis passing through the centre of gravity of the mass is cut by the same plane. Considering an element dm of this mass occupying any position E, and denoting OE by x, GE by z, and OG by ρ, the triangle OGE will give us the equation

Fig. 50



$$x^2 = z^2 + \rho^2 - 2\rho z \cos OGE.$$

Substituting in (1), and separating the terms, we have

$$mk^2 = \int z^2 dm + \int \rho^2 dm - 2\int \rho z dm \cos OGE,$$

or, since the distance ρ is constant, and $\int dm = m$,

$$mk^2 = \int z^2 dm + m\rho^2 - 2\rho \int dm . z \cos OGE.$$

But $z \cos OGE = GH =$ the lever arm of the mass dm with respect to the axis passing through the centre of gravity of the body. Hence

$$\int z dm \cos OGE$$

is the algebraic sum of the moments of all the particles of the body with respect to the axis passing through its centre of gravity; and this sum, by the principle of moments (No. 106) must be $= 0$, because the moment of their resultant is also $= 0$.

We have, therefore, simply

$$mk^2 = \int z^2 dm + m\rho^2.$$

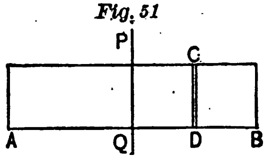
Now $\int z^2 dm$ is the moment of inertia of the mass m with respect to the axis passing through the centre of gravity. Denoting it by mr_0^2, we shall have

$$mk^2 = mr_0^2 + m\rho^2. \qquad (2)$$

Therefore, *the moment of inertia of a body with respect to any axis is equal to the moment of inertia with respect to a parallel axis through the centre of gravity of the body*, plus *the mass of the body into the square of the distance between the two axes.*

121. *Moment of inertia of a rectangle.* Let PQ (Fig. 51) be an axis passing through the centre of gravity of a rectangle, and lying in the plane of the rectangle perpendicularly to its length AB. Let m be the mass of the rectangle, and $AB = 2a$. The infinitesimal element $CD = dm$ placed at any distance x from the axis, will be found by the proportion

Fig. 51



$$2a : dx :: m : dm; \text{ or } dm = \frac{m}{2a}\, dx.$$

Substituting in (1), we have

$$mk^2 = \frac{m}{2a}\int x^2 dx,$$

and integrating from $x = -a$ to $x = a$, and reducing,

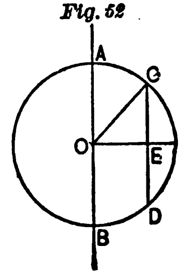
$$mk^2 = m\,\frac{a^2}{3}.$$

Hence the radius of gyration is here $k = \frac{a}{\sqrt{3}}$.

This result is independent of the altitude of the rectangle. Hence considering the straight line AB as a rectangle having an infinitesimal altitude, its moment of inertia will also be $m\,\frac{a^2}{3}$, m denoting the mass of the line.

122. *Moment of inertia of a circle, when the axis coincides with a diameter AB.*

Fig. 52



Let $OC = r$ (Fig. 52) be the radius of the circle, $CD = dm$ an element of its area parallel to the axis, $OE = x$ its distance from the centre O. We shall have

$$\pi r^2 : 2y\,dx :: m : dm, \text{ or } dm = \frac{2m}{\pi r^2}\,y\,dx.$$

Substituting in (1) and remembering that y is $= \sqrt{r^2 - x^2}$, we have

$$mk^2 = \frac{2m}{\pi r^2}\int (r^2 - x^2)^{\frac{1}{2}} x^2 dx.$$

By formula (33) we have

$$\int (r^2 - x^2)^{\frac{1}{2}} x^2 dx = -\frac{x}{4}(r^2 - x^2)^{\frac{3}{2}} + \frac{r^2}{4}\int (r^2 - x^2)^{\frac{1}{2}} dx,$$

and by formula (34),

$$\int (r^2 - x^2)^{\frac{1}{2}} dx = \tfrac{1}{2} \sqrt{r^2 - x^2} + \frac{r^2}{2} \sin^{-1} \frac{x}{r}.$$

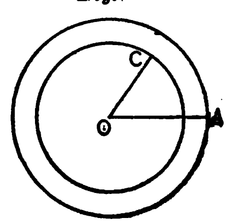
Substituting, and taking the integral from $x = -r$ to $x = r$, we have

$$mk^2 = m \frac{r^2}{4}.$$

Hence the radius of gyration is here $k = \frac{r}{2}$.

123. *Moment of inertia of a circle, when the axis through the centre is perpendicular to the plane of the circle.* With a radius $OC = x$ (Fig. 53) describe a circle, and give to its circumference a width dx. Then $2\pi x dx$ will represent an element of the area, and thus

Fig. 53



$$\pi r^2 : 2\pi x dx :: m : dm;$$

or

$$dm = \frac{2m}{r^2} x dx.$$

Substituting in (1), and integrating from $x = 0$ to $x = r$, we have

$$mk^2 = m \frac{r^2}{2}.$$

This formula is independent of the thickness of the circular plate; hence it will be true for any thickness. It therefore expresses the moment of inertia of a *solid cylinder* of any length, revolving about its axis.

124. *Moment of inertia of a circular ring with respect to an axis perpendicular to its plane.* Let r and r' be the extreme radii of the ring, and m its mass. Taking x between r and r', we have

$$\pi(r^2 - r'^2) : 2\pi x dx :: m : dm; \text{ and } dm = \frac{2m}{r^2 - r'^2} x dx.$$

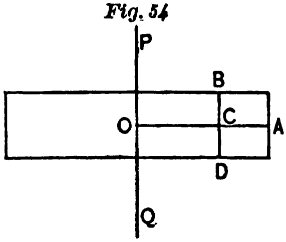
Substituting in (1), and integrating from $x = r'$ to $x = r$, we find

$$mk^2 = \frac{m}{2} \frac{r^4 - r'^4}{r^2 - r'^2} = m \frac{r^2 + r'^2}{2}.$$

This formula, being independent of the thickness of the ring, will be true for a *hollow cylinder* of any length.

125. *Moment of inertia of a cylinder with respect to an axis perpendicular to the axis of the cylinder.* Let the axis PQ (Fig. 54) be taken through the centre of gravity of the cylinder, and let BD be an element perpendicular to the axis of the cylinder, at a distance $OC = x$ from the axis of rotation. Let r be the radius of the cylinder, and μ the mass of the element BD. The moment of inertia of this element with respect to one of its diameters would be (No. 122) $= \mu \frac{r^2}{4}$; but its moment of inertia with respect to the axis PQ parallel to that diameter, and placed at a distance $OC = x$, will be $\mu\left(\frac{r^2}{4} + x^2\right)$, as we have shown (No. 120).

Fig. 54



Now, let $2a$ be the length, and m the mass, of the cylinder. We shall have

$$\mu : m :: dx : 2a, \text{ or } \mu = \frac{m}{2a}\, dx;$$

and therefore the moment of inertia of the whole cylinder will be found by integrating the expression $\frac{m}{2a}\left(\frac{r^2}{4} + x^2\right) dx$ between $x = -a$ and $x = a$. Integrating, and reducing, we shall find

$$mk^2 = m\left(\frac{r^2}{4} + \frac{a^2}{3}\right).$$

126. *Moment of inertia of a sphere.* Let the axis pass through the centre of the sphere. Let r be the radius of the sphere and μ the mass of an elementary segment perpendicular to the axis, placed at a distance x from the centre, and having a radius $y = \sqrt{r^2 - x^2}$. The moment of inertia of this element with respect to the axis is (No. 123)

$$\mu \frac{y^2}{2}, \text{ or } \frac{\mu}{2}(r^2 - x^2).$$

Now, if m be the mass of the sphere, we have

$$\mu : m :: \pi y^2 dx : \frac{4\pi r^3}{3};$$

whence

$$\mu = \frac{3m}{4r^3}\, y^2 dx;$$

and substituting this value of μ in the above expression, we shall have for the moment of inertia of the elementary segment

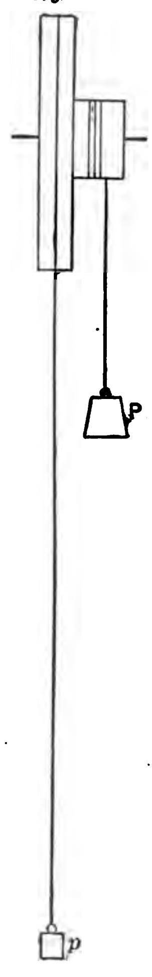
$$\frac{3m}{8r^3}(r^2 - x^2)^2\, dx.$$

Integrating this from $x = -r$ to $x = r$, we get, after reduction,

$$mk^2 = \frac{2mr^2}{5}$$

for the moment of inertia of the whole sphere.

Fig. 55



127. *Problem.* As an application of the theory of the moments of inertia, let R and r (Fig. 55) be the radii of two solid cylinders having the same horizontal axis, M being the mass of the larger and m that of the smaller; and let the weights p and P be applied to them respectively by a thread wrapped on their surfaces. If the two cylinders are so connected that they must rotate together, and if the weight p acts in a direction opposite to that of P, what will be the movement of these weights after a time t?

Solution. Adapting to our case the formula (No. 119)

$$\frac{d^2\vartheta}{dt^2} = \frac{Pr}{\int x^2 dm},$$

and reflecting that the moments of inertia of the two solid cylinders are $M\frac{R^2}{2}$ and $m\frac{r^2}{2}$ respectively, and that the moments of the accelerating forces are Pr and pR, we shall have

$$\frac{d^2\vartheta}{dt^2}=\frac{Pr-Rp}{\frac{MR^2}{2}-\frac{mr^2}{2}}=2\,\frac{Pr-Rp}{MR^2-mr^2}.$$

Let now M' and m' be the masses of the weights P and p respectively. Then g being the action of gravity, we have $P=M'g$, $p=m'g$. And therefore

$$\frac{d^2\vartheta}{dt^2}=2g\,\frac{M'r-Rm'}{MR^2-mr^2};$$

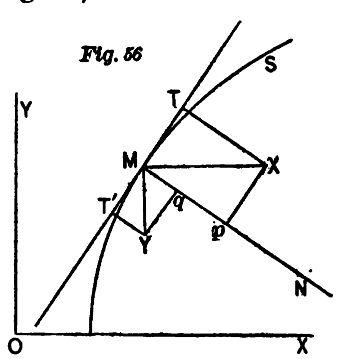
and integrating from $t=0$ to $t=t$,

$$\frac{d\vartheta}{dt}=2gt\,\frac{M'r-Rm'}{MR^2-mr^2},\quad \vartheta=gt^2\,\frac{M'r-Rm'}{MR^2-mr^2}.$$

Curvilinear Movement.

128. A free point M (Fig. 56) cannot move in a curve, unless its direction be continually changed by an action proceeding from some other direction. Let us conceive this action decomposed into two, X and Y, respectively parallel to the co-ordinate axes OX and OY. Draw MN normal, and MT tangent to the curve at the point M. Resolve X into MT

Fig. 56



tangential, and Mp normal, also Y into MT' tangential, and Mq normal to the curve. Then the forces X and Y may be replaced by a *tangential* force $T = MT - MT'$, and by a *normal*, or *centripetal* force $N = Mp + Mq$.

Now, calling ϑ the angle that the element ds of the curve at M makes with the axis of abscissas, we have

$$MT = X\cos\vartheta,\ MT' = Y\sin\vartheta,$$

$$Mp = X\sin\vartheta,\ Mq = Y\cos\vartheta.$$

Hence

$$T = X\cos\vartheta - Y\sin\vartheta,$$

$$N = X\sin\vartheta + Y\cos\vartheta.$$

But, according to our usual notation,

$$X = \frac{d^2x}{dt^2}, \quad Y = -\frac{d^2y}{dt^2},$$

$$\sin\vartheta = \frac{dy}{ds}, \quad \cos\vartheta = \frac{dx}{ds};$$

Therefore

$$T = \frac{d^2x}{dt^2}\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{dy}{ds};$$

$$N = \frac{d^2x}{dt^2}\frac{dy}{ds} - \frac{d^2y}{dt^2}\frac{dx}{ds}.$$

The first of these equations can be reduced to

$$T = \frac{d\,(dx^2 + dy^2)}{dt^2.2ds} = \frac{d\,(ds^2)}{dt^2.2ds} = \frac{2ds d^2s}{2ds dt^2} = \frac{d^2s}{dt^2}.$$

The second, being multiplied and divided by ds^2, becomes

$$N = -\frac{ds^2}{dt^2}\cdot\frac{d^2ydx - d^2xdy}{ds^3};$$

and this, according to the remark made by us on the expression for the radius of curvature when t is the independent variable (No. 46), will become

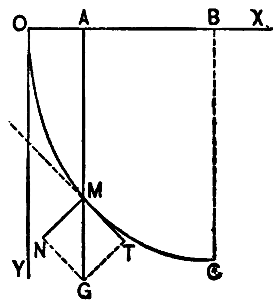
$$N = -\frac{ds^2}{dt^2}\cdot\frac{1}{R} = -\frac{v^2}{R}.$$

Such is the expression of the *centripetal* acceleration. Hence *the centripetal force is equal to the product of the mass into the square of the tangential velocity divided by the radius of curvature*

The moving point, while obeying the centripetal action, always keeps its tendency to follow a straight line, that is, the tangential direction, and thus to recede from the centre of curvature. This centrifugal tendency, so far as counteracted and thwarted by the centripetal action, is usually called *the centrifugal force*, and its intensity is measured by that of the action by which it is thwarted. Hence the centrifugal force is equal to the centripetal, and directly opposed to it. The centripetal and centrifugal forces are commonly called *central forces.*

129. Let a point M roll down a curve OMC (Fig. 57) under the action of gravity. Let OX and OY be the cö-ordinate axes, and let the ordinates downward be positive. When the moving point has reached any position M, the action of gravity MG, by which its movement is accelerated, may be decomposed into MN normal, and MT tangential to the curve. The first will be destroyed by the resistance of the curve, which we assume to be invariably fixed, and which is thus playing the part of a centripetal force. The second will have its whole effect. Let ϑ be the angle that the curve at M makes with the axis OX. Then

Fig. 57



$$MT = g \sin \vartheta = g \frac{dy}{ds},$$

or

$$\frac{d^2s}{dt^2} = g \frac{dy}{ds},$$

Hence

$$\frac{2ds}{dt} \frac{d^2s}{dt} = 2gdy, \text{ and } \left(\frac{ds}{dt}\right)^2 = 2gy.$$

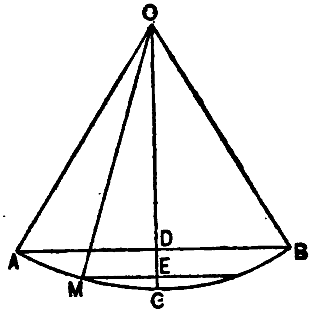
This velocity is the velocity due to the height y (No. 100). Therefore the velocity acquired by a body rolling down a curve under the action of gravity, is equal to that which it acquires by falling freely through the same vertical height.

This result is true not only when g is constant, but also when g varies according to a fixed law; for, even in this case, g may be regarded as constant from element to element, inasmuch as the same law of variation applies to the elements of the curve and to those of the vertical line. Hence a body falling toward the sun on a spiral line will have the same final velocity as though it had fallen directly towards its centre.

130. *The simple pendulum.* A material point

A (Fig. 58) suspended from a horizontal axis by a rigid line AO without weight, and free to oscillate about that axis, constitutes a simple pendulum.

Fig. 58



Let $AO = l$, the angle $AOC = \alpha$, and the angle $MOC = \vartheta$. When the point A under the action of gravity reaches the point M, it will have acquired a velocity $\frac{ds}{dt} = \sqrt{2gy}$ (No. 129), y being $= DE$. But

$$ds = l d\vartheta, \text{ and } y = OE - OD = l(\cos\vartheta - \cos\alpha);$$

therefore

$$l\frac{d\vartheta}{dt} = \sqrt{2gl(\cos\vartheta - \cos\alpha)};$$

whence

$$dt = \sqrt{\frac{l}{2g}} \cdot \frac{d\vartheta}{\sqrt{\cos\vartheta - \cos\alpha}}.$$

From Maclaurin's formula we have

$$\cos\vartheta = 1 - \frac{\vartheta^2}{2} + \frac{\vartheta^4}{2.3.4} - \ldots$$

$$\cos\alpha = 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{2.3.4} - \ldots;$$

hence, if the arc AC be small enough to allow us to neglect all the terms of the series after the second, we shall have $\cos\vartheta - \cos\alpha = \frac{1}{2}(\alpha^2 - \vartheta^2)$, and

$$dt = \sqrt{\frac{l}{g}} \cdot \frac{d\vartheta}{\sqrt{a^2 - \vartheta^2}}.$$

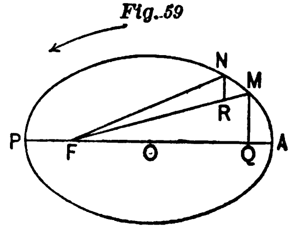
Integrating from $\vartheta = -a$ to $\vartheta = a$, we shall have

$$t = \pi \sqrt{\frac{l}{g}}.$$

This is the time of one excursion, or semi-oscillation, of a simple pendulum, when the amplitude of the excursion is small; that is, not exceeding 9 or 10 degrees.

131. *Planetary orbits.* By the first of Kepler's laws, the orbits of planets are ellipses, of which one focus is occupied by the sun.

Fig. 59



Let PMA (Fig. 59) be a planetary orbit, and F the focus occupied by the sun. Let the planet, at a given instant dt, be moving from M towards the perihelion P. Make $FM = \rho$, the angle

$$MFA = v,\ OQ = x,\ MQ = y,\ OF = c,$$

and let φ be the solar action at the unit of distance. We shall have, from the inspection of the figure,

$$FQ = \rho \cos v = FO + OQ = c + x,$$

$$MQ = \rho \sin v = y,$$

whence

$$\cos v = \frac{c + x}{\rho}, \quad \sin v = \frac{y}{\rho}.$$

The action of the sun at a distance ρ being $\frac{\varphi}{\rho^2}$, we have also

$$\frac{d^2x}{dt^2} = -\frac{\varphi}{\rho^2}\cos v = -\frac{\varphi(c+x)}{\rho^3},$$

$$\frac{d^2y}{dt^2} = -\frac{\varphi}{\rho^2}\sin v = -\frac{\varphi y}{\rho^3}.$$

Multiplying the first of these equations by dx, and the second by dy, adding the products, and remarking that we have

$$\rho = \sqrt{(c+x)^2+y^2},$$

we find

$$\frac{dx}{dt}\frac{d^2x}{dt} + \frac{dy}{dt}\frac{d^2y}{dt} = -\varphi\frac{(c+x)\,dx + ydy}{\sqrt{\{(c+x)^2+y^2\}^3}};$$

and integrating,

$$\frac{dx^2+dy^2}{dt^2} = \frac{2\varphi}{\sqrt{(c+x)^2+y^2}} + C = \frac{2\varphi}{\rho} + C,$$

or, since $dx^2 + dy^2 = \overline{MN}^2 = ds^2$,

$$\left(\frac{ds}{dt}\right)^2 = \frac{2\varphi}{\rho} + C \qquad (1)$$

Calling V' the velocity of the planet at the perihelion P, and making $PF = \rho'$, we have

$$C = V'^2 - \frac{2\varphi}{\rho'}.$$

Therefore

$$\left(\frac{ds}{dt}\right)^2 = V'^2 - 2\varphi\left(\frac{1}{\rho'} - \frac{1}{\rho}\right). \qquad (2)$$

Calling V'' the velocity of the planet at the aphelion A, and making $FA = \rho''$, we have

$$C = V''^2 - \frac{2\varphi}{\rho''}.$$

Therefore

$$\left(\frac{ds}{dt}\right)^2 = V''^2 + 2\varphi\left(\frac{1}{\rho} - \frac{1}{\rho''}\right); \qquad (3)$$

and eliminating $\frac{ds}{dt}$ between (2) and (3), we obtain

$$V''^2 = V'^2 - 2\varphi\left(\frac{1}{\rho'} - \frac{1}{\rho''}\right). \qquad (4)$$

Now, by the second of Kepler's laws, the areas described by the radius vector are proportional to the times in which they are described. Hence two areas described in a second by the planet must be equal; and therefore $\frac{1}{2}V'\rho' = \frac{1}{2}V''\rho''$, whence

$$V' = V''\frac{\rho''}{\rho'}.$$

This value of V' substituted in (4) will lead to

$$V''^2 = 2\varphi\frac{1}{\rho''^2}\cdot\frac{\rho'\rho''}{\rho'+\rho''}. \qquad (5)$$

But, if a and b be the major and minor semi-axes of the orbit, we have

$$\rho' + \rho'' = 2a, \quad \rho'\rho'' = (a-c)(a+c) = a^2 - c^2 = b^2.$$

Substituting in (5),

$$V''^2 = \frac{\varphi}{\rho''^2}\cdot\frac{b^2}{a}. \qquad (6)$$

Comparing equations (3) and (5), we obtain

$$\left(\frac{ds}{dt}\right)^2 = 2\varphi \left\{\frac{\rho'}{\rho''(\rho'+\rho'')} + \frac{1}{\rho} - \frac{1}{\rho''}\right\},$$

which, owing to (6), will be reduced to

$$\left(\frac{ds}{dt}\right)^2 = 2\varphi\left(\frac{1}{\rho} - \frac{1}{2a}\right) = \frac{\varphi}{\rho^2} \cdot \frac{2a\rho - \rho^2}{a}. \quad (7)$$

And now, from the general equation

$$ds^2 = (\rho dv)^2 + d\rho^2$$

we have

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{\rho dv}{dt}\right)^2 + \left(\frac{d\rho}{dt}\right)^2 \quad (8)$$

from which we shall easily eliminate $\frac{\rho dv}{dt}$ by considering that, since the area described by the radius vector in the time dt is $\frac{1}{2}\rho^2 dv$, the area described in the unit of time will be $\frac{1}{2}\rho^2 \frac{dv}{dt}$. But we have already found that this area can be expressed by $\frac{1}{2} V''\rho''$. Hence

$$\rho^2 \frac{dv}{dt} = V''\rho'', \quad \rho \frac{dv}{dt} = \frac{V''\rho''}{\rho},$$

$$\left(\frac{\rho dv}{dt}\right)^2 = \frac{V''^2\rho''^2}{\rho^2},$$

or substituting the value of V'' from equation (6), and reducing,

$$\left(\frac{\rho dv}{dt}\right)^2 = \frac{\varphi}{\rho^2} \cdot \frac{b^2}{a}.$$

This being substituted in (8), we have

$$\left(\frac{ds}{dt}\right)^2 = \frac{\varphi}{\rho^2} \cdot \frac{b^2}{a} + \left(\frac{d\rho}{dt}\right)^2.$$

Now, eliminating $\frac{ds}{dt}$ between this equation, and (7), we shall find

$$\left(\frac{d\rho}{dt}\right)^2 = \frac{\varphi}{a} \cdot \frac{2a\rho - \rho^2 - b^2}{\rho^2},$$

whence

$$dt = -\sqrt{\frac{a}{\varphi}} \cdot \frac{\rho d\rho}{\sqrt{2a\rho - \rho^2 - b^2}};$$

or, since $b^2 = a^2 - c^2$,

$$dt = -\sqrt{\frac{a}{\varphi}} \cdot \frac{\rho d\rho}{\sqrt{c^2 - (\rho - a)^2}} = -\sqrt{\frac{a}{\varphi}} \cdot \frac{(\rho - a + a)\, d\rho}{\sqrt{c^2 - (\rho - a)^2}}$$

$$= -\sqrt{\frac{a}{\varphi}} \left\{ \frac{a d\rho}{\sqrt{c^2 - (\rho - a)^2}} + \frac{(\rho - a)\, d\rho}{\sqrt{c^2 - (\rho - a)^2}} \right\},$$

of which the integral is

$$t = \sqrt{\frac{a}{\varphi}} \left(a \cos^{-1} \frac{\rho - a}{c} + c \sqrt{1 - \left(\frac{\rho - a}{c}\right)^2} \right) + C.$$

If we reckon the time from the aphelion A, we shall have $t = 0$ when $\rho = a + c$, that is, when $\rho - a = c$; and then $C = 0$. Therefore

$$t = \sqrt{\frac{a}{\varphi}} \left(a \cos^{-1} \frac{\rho - a}{c} + c \sqrt{1 - \left(\frac{\rho - a}{c}\right)^2} \right). \quad (9)$$

Such is the expression for the time taken by the planet in measuring any portion AM of its orbit, ρ being the radius vector of the point M.

The time T employed in measuring the semi-orbit $AM'P$ will be found by taking

$$\rho = a - c, \text{ or } \rho - a = -c.$$

Then (9) gives

$$T = a\pi \sqrt{\frac{a}{\varphi}}. \qquad (10)$$

This equation shows that the squares of the times of the revolutions of two planets are to each other directly as the cubes of the transverse axes of their orbits. This is the third among Kepler's laws.

The same equation shows also that the time of a revolution is independent of the minor axis b of the orbit. Assume b so small, that the ellipse may sensibly be reduced to a double straight line. Then the focus will sensibly coincide with the extremity P of the transverse axis, and T would be the time taken by the planet in falling directly upon the sun from the distance $AP = 2a$.

From (10) we have $\sqrt{\frac{a}{\varphi}} = \frac{T}{a\pi}$. Hence (9) may be transformed into

$$t = T\left\{\frac{1}{\pi}\cos^{-1}\frac{\rho - a}{c} + \frac{c}{a\pi}\sqrt{1 - \left(\frac{\rho - a}{c}\right)^2}\right\},$$

and if e be the eccentricity of the orbit, we may substitute e for $\frac{c}{a}$, and reduce the equation to the form

$$t = T\left\{\frac{1}{\pi}\cos^{-1}\left[\frac{\frac{\rho}{a} - 1}{e}\right] + \frac{e}{\pi}\sqrt{1 - \left[\frac{\frac{\rho}{a} - 1}{e}\right]^2}\right\}.$$

Lastly, making

$$\frac{\frac{\rho}{a}-1}{e}=\cos\vartheta \tag{11}$$

the equation will take the form

$$t=T\left(\frac{\vartheta+e\sin\vartheta}{\pi}\right). \tag{12}$$

The formulas (11) and (12), and the polar equation of the ellipse, which is

$$\frac{\rho}{a}=\frac{1-e^2}{1-e\cos v} \tag{13}$$

will suffice to determine the time taken by the planet in measuring any given angle v reckoned from the aphelion, when the eccentricity of the orbit is known.

Scholium. The eccentricity of the terrestrial orbit being $e=0.016833$, and the earth, during the tropical year, measuring only 359° 59′ 9″.8 around the sun, the longitude of the aphelion (which is reckoned from the vernal equinox, and which on the 1st of January, 1800, was 99° 30′ 8″.39) is increasing every year by 50″.2. The length of the tropical year is $=365^d.242256$. Will the student, with these data, and with the aid of the last three formulas, try to determine the length of the four seasons for some given year?